Sobolev Spaces

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206D: Finite Element Methods
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Basic Definitions

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A function $u \in L^2$ has as its **weak derivative** $v = \mathcal{D}_{\boldsymbol{\alpha}} u = \partial^{\boldsymbol{\alpha}} u$ if

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C^∞ is the space of all infinitely continuously differentiable functions.
The $C_0^\infty \subset C^\infty$ are all functions zero outside a compact set.

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We refer to H^m with this inner-product as a **Sobolev space**. Also denoted by $W^{m,2}$.

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Let $\Omega \subset \mathbb{R}^n$ be an open set with piecewise smooth boundary. Let $m \geq 0$, then $C^\infty(\Omega) \bigcap H^m(\Omega)$ is dense in $H^m(\Omega)$ under the norm $\|\cdot\|_m$.

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We have the following relations between the function spaces

$$\begin{array}{ccccccccccc}
L^2(\Omega) & = & H^0(\Omega) & \supset & H^1(\Omega) & \supset & H^2(\Omega) & \cdots & \supset & H^m(\Omega) \\
 & & \shortparallel & & \cup & & \cup & & & \cup \\
 & = & H_0^0(\Omega) & \supset & H_0^1(\Omega) & \supset & H_0^2(\Omega) & \cdots & \supset & H_0^m(\Omega).
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Similarly, we obtain $W_0^{m,p}$ by completing $C_0^\infty(\Omega) \subset L^p(\Omega)$ under $\|\cdot\|_m$.

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Consider a given domain Ω and compact sets $K \subset \Omega$. We define the set of **locally integrable** functions as

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These functions can behave poorly near the boundary of Ω as illustrated by $v(x) = \phi(1/\text{dist}(x, \partial\Omega))$ where $\phi(x) = e^{e^x}$ which still yields $v \in L^1_{\text{loc}}(\Omega)$.

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Example: Let $f(x) = 3$ on the rationals $\mathbb{Q}$ and $f(x) = 2$ on the positive irrationals $\mathbb{R}^+ \setminus \mathbb{Q}$ and $f(x) = -1$ on the negative irrationals $\mathbb{R}^- \setminus \mathbb{Q}$.

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Poincaré-Friedrichs Inequality

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Poincaré-Friedrichs Inequality: Consider domain $\Omega \subset Q = [0, s]^n$, Q is cube of side-length s. Then

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This shows the 1-semi-norm bounds the 0-norm.

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Proof: Since $v \in H_0^1$ and using a point on the boundary $(0, x_2, x_3, \ldots, x_n)$ we can express v as

$$v(x_1, x_2, \ldots, x_n) \quad = \quad v(0, x_2, \ldots, x_n) + \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz = \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz$$

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$$|v(\mathbf{x})|^2 \leq \left(\int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz \right)^2 \quad \leq \quad \int_0^{x_1} 1^2 dz \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz$$

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We integrate over the cube $Q = [0,s]^n$ with v, $\partial^1 v$ extended to vanish outside of Ω.

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We now integrate both sides over $\int_0^s$ and note RHS independent of x_1

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$$\begin{aligned} |v(\mathbf{x})|^2 \leq \left(\int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz\right)^2 &\leq \int_0^{x_1} 1^2 dz \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz \\ &\leq s \int_0^s |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz \end{aligned}$$

We now integrate both sides over $\int_0^s$ and note RHS independent of x_1

$$\int_0^s |v(\mathbf{x})|^2 dx_1 \leq s^2 \int_0^s |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz$$

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Definition

Trace-Free Sobolev Spaces: We denote by $\mathring{W}_p^1(\Omega)$ the subset of $W_p^1(\Omega)$ consisting of the functions whose trace on the boundary $v|_{\partial\Omega}$ is zero. In particular,

$$\mathring{W}_p^1(\Omega) = \{v \in W_p^1(\Omega) \mid v|_{\partial\Omega} = 0 \text{ in } L^2(\partial\Omega)\}.$$

Similarly, $\mathring{W}_p^k(\Omega)$ consists of functions whose derivatives of order $k-1$ are in $\mathring{W}_p^1(\Omega)$, so that

$$\mathring{W}_p^k(\Omega) = \{v \in W_p^k(\Omega) \mid v^{(\alpha)}|_{\partial\Omega} = 0 \text{ in } L^2(\partial\Omega), \;\forall|\alpha| < k\}.$$