Exercises

ODEs and Dynamical Systems — Paul J. Atzberger
MATH 243A — `http://atzberger.org/`

1. Consider the Lipschitz function $g(x)$ and continuous function $f(x)$ for the ODE

$$\dot{x} = g(x), \quad \dot{y} = f(x)y.$$

Show for a given initial condition x_0, y_0 if a solution exists on an interval then there is at most one solution. (*Hint:* Use Gronwall's Inequality).

2. Consider the ODE

$$\dot{x} = x^{2/3}.$$

   (a) Show that $x^{2/3}$ is not a Lipschitz function on any interval $[0, b]$ with $b > 0$.

   (b) Show there are infinitely many solutions satisfying the initial value problem when $x(0) = 0$.

3. For the ODE $\dot{x} = f(x)$ consider the Forward-Euler Method $w_{k+1} = w_k + hf(w_k)$. Show for the ODE $\dot{x} = -\lambda x$ with $\lambda > 0$ that the Forward-Euler Method is only stable provided that $0 < h < 2/\lambda$. By stability here, we mean that when $\lim_{t\to\infty} x(t) = 0$ we have that $\lim_{k\to\infty} w_k = 0$. If we were to modify the numerical approximation to be the Implicit-Euler Method with $w_{k+1} = w_k + hf(w_{k+1})$ for $\dot{x} = -\lambda x$ what do the conditions for h for stability become?

4. Consider the ODE $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $f_1(x_1, x_2) = x_2$ and $f_2(x_1, x_2) = \mu(1 - x_1^2)x_2 - x_1$. This is the Van der Pol oscillator. Integrate the dynamics of this ODE using the Runge-Kutta Methods below

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{array}{c|cc} 0 & 1/4 & -1/4 \\ 2/3 & 1/4 & 5/12 \\ \hline & 1/4 & 3/4 \end{array} \qquad \begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

   (a) Consider the cases when $\mu = 0, 2, 4$ and initial conditions $\mathbf{x}_0 = (-3, -3)$, $\mathbf{x}_0 = (-3, 3)$ and $\mathbf{x}_0 = (1, 1)$.

   (b) Plot the 2D trajectories obtain for each of the methods along with the direction field.

   (c) Discuss the behaviors of the different methods.

5. Consider the mechanical system with dynamics $m\ddot{\mathbf{x}} = F(\mathbf{x})$. This can also be expressed as the system $\dot{\mathbf{x}} = \mathbf{v}$, $\dot{\mathbf{v}} = F(\mathbf{x})$. The Velocity-Verlet Method is given by

$$\begin{aligned} w_{2,k+\frac{1}{2}} &= w_{2,k} + \frac{1}{2}hf(w_{1,k}) \\ w_{1,k+1} &= w_{1,k} + hw_{2,k+\frac{1}{2}} \\ w_{2,k+1} &= w_{2,k+\frac{1}{2}} + \frac{1}{2}hf(w_{1,k+1}). \end{aligned}$$

In the notation, $\mathbf{w} = (w_1, w_2) \approx (x, \dot{x}) = (x, v)$.

(a) Consider the mechanical system when $F(\mathbf{x}) = -k_0\mathbf{x}$ with $k_0 = 1, m = 1$ and $\mathbf{x} \in \mathbb{R}^1$. Derive the exact analytic solution when $\mathbf{x}(0) = 1$ and $\dot{\mathbf{x}}(0) = 0$.

(b) Numerically approximate the solution using the Velocity-Verlet Method, Forward-Euler, and RK4. Compare your results for each of the methods when taking $h = 1, 10^{-1}, 10^{-2}$ at time $T = 3\pi$.

(c) Plot the error vs time for each of the methods for $t \in [0, T]$ with $T \gg 2\pi$.

(d) Discuss the behaviors of these different methods for approximating the dynamics of the mechanical system.