Mixed Methods

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206D: Finite Element Methods
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When $\mathcal{L}$ contains only bilinear and quadratic expressions in u and λ, we obtain a saddle point problem.

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The conditions (i) and (ii) alone imply that L is *isomorphism* on W^0 where

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Remark: When this criteria holds for the spaces U_h, V_h, we say they satisfy the Babuska-Brezzi Condition.

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$$\begin{aligned} &B : X \rightarrow M', && B' : M \rightarrow X' \\ &\langle Bu, \mu\rangle = b(u,\mu),\ \ \forall \mu \in M, && \langle B'\lambda, v\rangle = b(v,\lambda),\ \ \forall v \in X. \end{aligned}$$

The Saddle Point Problem I can be expressed as

Saddle Point Problem II

Find $(u,\lambda) \in X \times M$ satisfying

$$\begin{aligned} Au + B'\lambda &= f, \\ Bu &= g. \end{aligned}$$

Inf-Sup Lemma

Saddle Point Problems

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Remark:

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Remark: Note the coercivity is assumed only for v in kernel of B (see def. of V).

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$$\Delta u = -f,\ x \in \Omega, \quad u = 0,\ x \in \Gamma_0, \quad \nabla u \cdot \mathbf{n} = 0,\ x \in \Gamma_1.$$

We use that $\Delta u = \operatorname{div} \operatorname{grad} u$. Let $\sigma = \operatorname{grad} u$, then the Poisson problem becomes

$$\begin{aligned} \operatorname{grad} u &= \sigma \\ \operatorname{div} \sigma &= -f \end{aligned}$$

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Poisson Problem: Mixed Formulation

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$$(\sigma, \tau)_{0,\Omega} - (\tau, \nabla u)_{0,\Omega} = 0,\ \forall \tau \in L_2(\Omega)^d$$

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Poisson Problem: Saddle-Point Formulation

Poisson Problem: Mixed Methods

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Poisson Problem: Mixed Methods

Poisson Problem: Mixed Formulation

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Poisson Problem: Mixed Methods

Poisson Problem: Mixed Formulation

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$$a(\sigma,\tau) - b(\tau, v) = 0$$

Poisson Problem: Mixed Methods

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This establishes stability of the formulation.

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We can obtain stable Finite Element discretizations for triangulations $\mathcal{T}_h$.

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Poisson Problem: Stable Mixed Finite Element Spaces

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$$X_h := \left(\mathcal{M}^{k-1}\right)^d = \{\sigma_h \in L_2(\Omega)^d; \sigma_h|_T \in \mathcal{P}_{k-1}, \; \forall T \in \mathcal{T}_h\}$$

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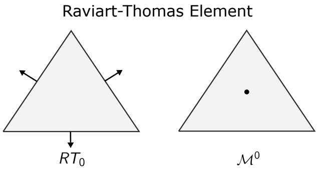
Poisson Problem: Mixed Methods

Raviart-Thomas Element

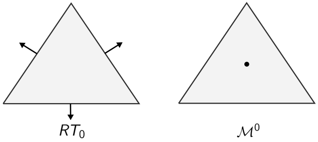


Poisson Problem: Mixed Methods

Raviart-Thomas Element

$$X_h \quad := \quad RT_k := \left\{ \tau \in L_2(\Omega)^2;\ \tau|_T = \begin{pmatrix} a_T \\ b_T \end{pmatrix} + c_T \begin{pmatrix} x \\ y \end{pmatrix},\ a_T, b_T, c_T \in \mathcal{P}_k,\ \forall T \in \mathcal{T}_h, \tau \cdot n \in \tilde{C}(\partial T) \right\}$$

Raviart-Thomas Element



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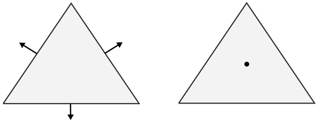

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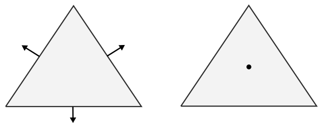
The $\tau \cdot n \in \tilde{C}(\partial T)$ denotes that $\tau \cdot n$ is continuous on the inter-element boundaries.



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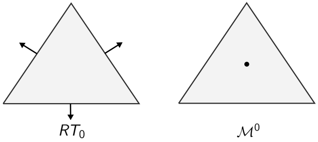
These can be shown to satisfy the Inf-Sup Condition for the Poisson Problem Mixed Formulation.

Raviart-Thomas Element



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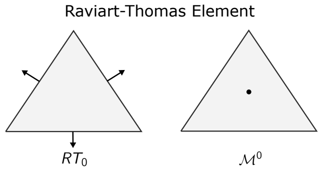
For $k = 0$, $p \in (\mathcal{P}_1)^2$ has



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These can be shown to satisfy the Inf-Sup Condition for the Poisson Problem Mixed Formulation.

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$$p(x,y) = \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} x \\ y \end{pmatrix}.$$

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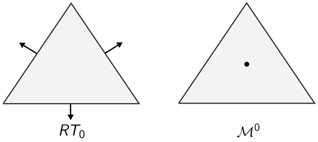
Raviart-Thomas Element



Poisson Problem: Mixed Methods

Raviart-Thomas Element

$$X_h := RT_k := \left\{\tau \in L_2(\Omega)^2;\ \tau|_T = \begin{pmatrix} a_T \\ b_T \end{pmatrix} + c_T \begin{pmatrix} x \\ y \end{pmatrix},\ a_T, b_T, c_T \in \mathcal{P}_k,\ \forall T \in \mathcal{T}_h, \tau \cdot n \in \tilde{C}(\partial T)\right\}$$

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The $\tau \cdot n \in \tilde{C}(\partial T)$ denotes that $\tau \cdot n$ is continuous on the inter-element boundaries.

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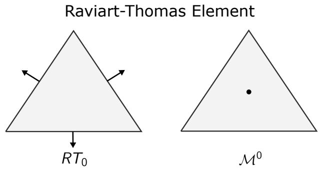

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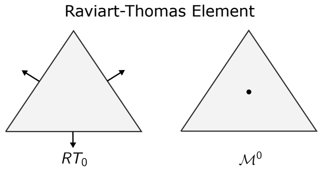

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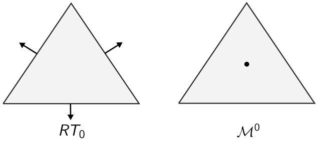
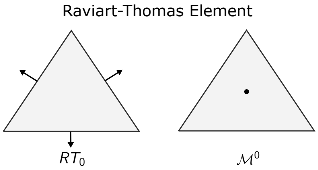
$$\left(T, (\mathcal{P}_0)^2 + \mathbf{x} \cdot \mathcal{P}_0,\ n_i \cdot p(z_i), i = 1,2,3,\ z_i \text{ is edge midpoint.} \right)$$

Raviart-Thomas Element



Poisson Problem: Raviart-Thomas Element

Mesh-Dependent Norms:



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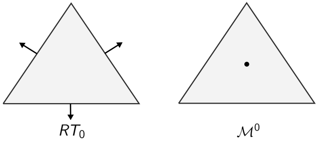

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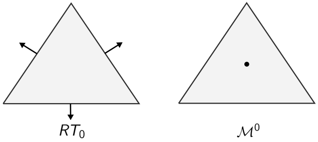

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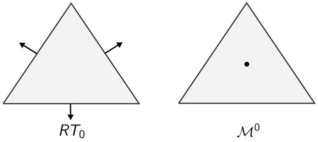

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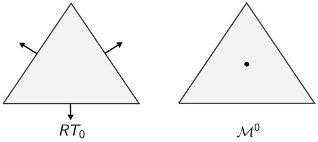

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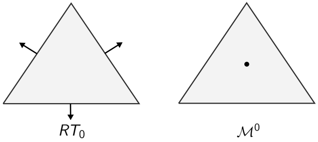

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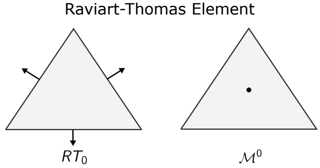

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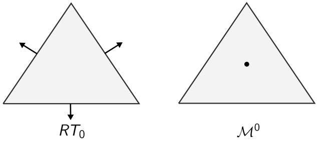

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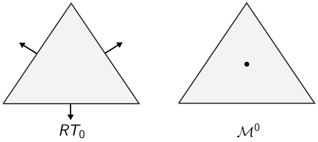

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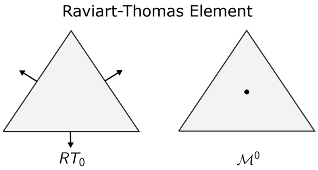

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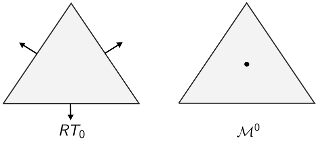

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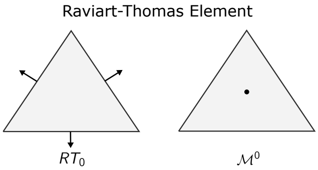

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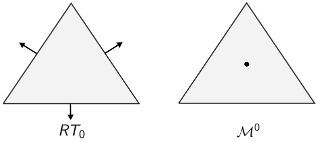
Inf-Sup Condition must be established.

Raviart-Thomas Element



Poisson Problem: Raviart-Thomas Element

Lemma: The Inf-Sup Condition

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where $\beta > 0$ and depends on k

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on each edge $e \subset \Gamma_h$. Since in this case the area term in Green's Identity for b vanishes, we have

$$b(\tau, v) = h^{-1} \int_{\Gamma_h} |J(v)|^2 ds = ch^{-1} \sum_{e \subset \Gamma_h} \|J(v)\|_{0,e}^2 = |v|_{1,h}^2.$$

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The bilinear form b with the RT-elements satisfies

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(By Theorem I): For a $q \in L_{2,0}$, exists $v \in H_0^1(\Omega)^n$ satisfying $\operatorname{div} v = q$ and $\|v\|_{1,\Omega} \leq c\|q\|_{0,\Omega}$ (from previous thm.)

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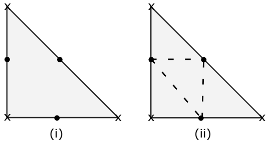
Stokes Hydrodynamic Equations: Taylor-Hood Element

Consider triangulation $\mathcal{T}_h$ and polymomial shape spaces $\mathcal{P}_j$.

Taylor-Hood Element



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Taylor-Hood Elements: Stability achieved by velocity field in polynomial space larger degree than the pressure space.

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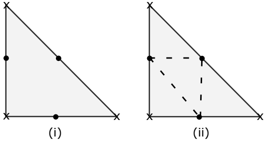

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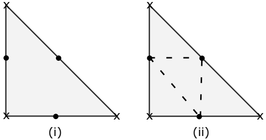

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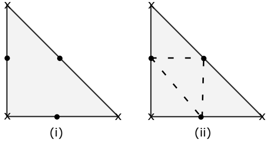

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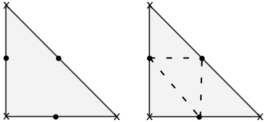

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Modified Taylor-Hood Element: Use piece-wise linear functions on sub-triangles (macro element)

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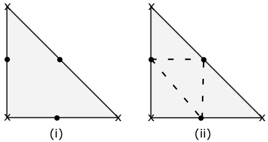

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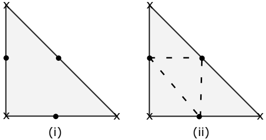

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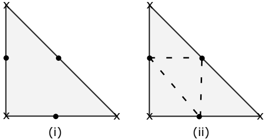

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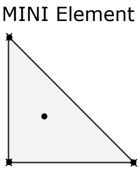
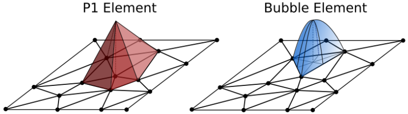
Figure: x denotes pressure values, $\cdot$ denotes velocity values.

Stokes Hydrodynamic Equations: MINI Element



MINI Elements: Achieves stability by using interior "bubble" elements.



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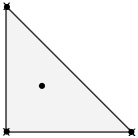

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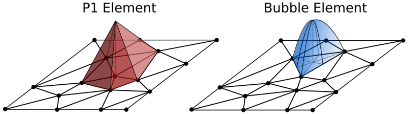

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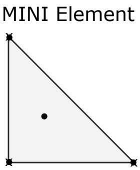

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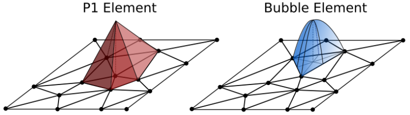
Note, b vanishes on boundary of T.



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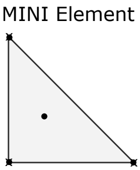

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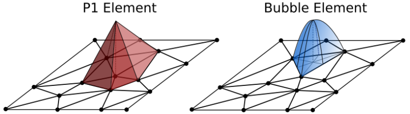

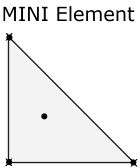

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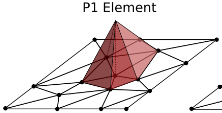
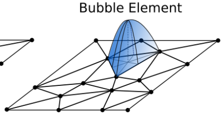
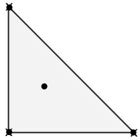
where $B_3 := \{v \in C^0(\bar{\Omega});\ v|_T \in \text{span}[\lambda_1\lambda_2\lambda_3],\ \forall T \in \mathcal{T}_h\}$.





Stokes Hydrodynamic Equations: MINI Element

MINI Element



MINI Elements: Achieves stability by using interior "bubble" elements.

For triangle, let $\lambda_1, \lambda_2, \lambda_3$ denotes the **barycentric coordinates** of a points $\mathbf{x}$.

Add to the shape space the "bubble" function

$$b(\mathbf{x}) = \lambda_1 \lambda_2 \lambda_3.$$

Note, b vanishes on boundary of T.
The finite element spaces are

$$X_h := \left[\mathcal{M}^1_{0,0} \oplus B_3\right]^2, \quad M_h := \mathcal{M}^1_0 \bigcap L_{2,0}(\Omega),$$

where $B_3 := \{v \in C^0(\bar{\Omega});\ v|_T \in \text{span}[\lambda_1\lambda_2\lambda_3],\ \forall T \in \mathcal{T}_h\}$.

Figure: x denotes pressure values, $\cdot$ denotes velocity values.