Introduction to Machine Learning

Foundations and Applications

Paul J. Atzberger
University of California Santa Barbara

Statistical Learning Theory
PAC-Learning
Generalization Bounds

Statistical Learning Theory

“There is nothing more practical than a good theory.”
-- James C. Maxwell.

Framework for characterizing learning problems and algorithms.

Goal: Assess how well a model predicts future input-output relations.


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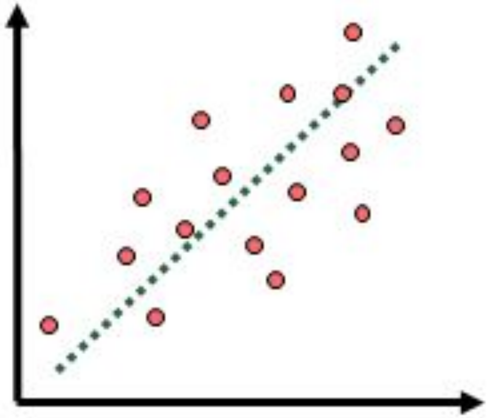

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Mathematical Definitions: Consider c: $X \rightarrow Y$, X-input, Y-output.
Let c = concept, $\mathcal{C}$ = {concept class}, $\mathcal{H}$ = {hypothesis function space},
D$_{x,y}$ ~ $X \times Y$ be unknown probability distribution on $X \times Y$, and
$V(h(x_j), y_j)$ = loss function.

Learning Problem: Find the best $h \in \mathcal{H}$ so that $E_D[V(\mathrm{h}(x), y)]$ is minimized when c $\in C$, $\mathrm{y} = c(x)$.

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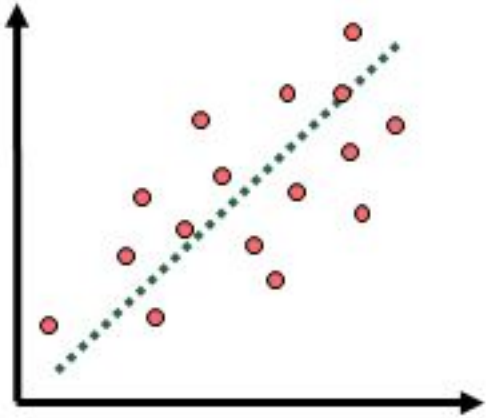

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Classification: $V(h(x), y) = I_{h(x) \neq y}$, (zero-one loss).

Regression: $V(h(x), y) = (h(x) - y)^2$, (least-squares L^2 -loss).

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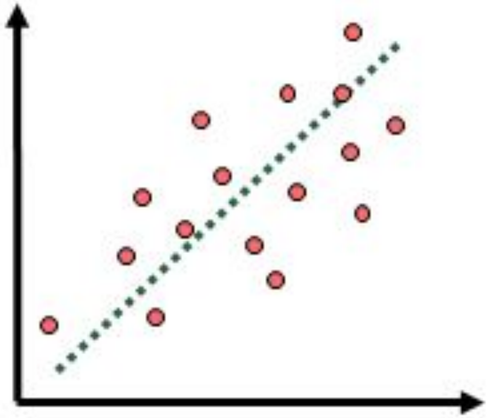

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Important to learning, the choice of hypothesis class H and loss used!


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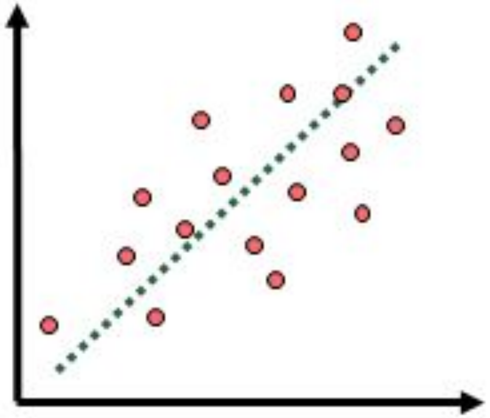

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Practical Challenges: Distribution D usually unknown, optimization is often non-convex and in high-dimensional spaces.

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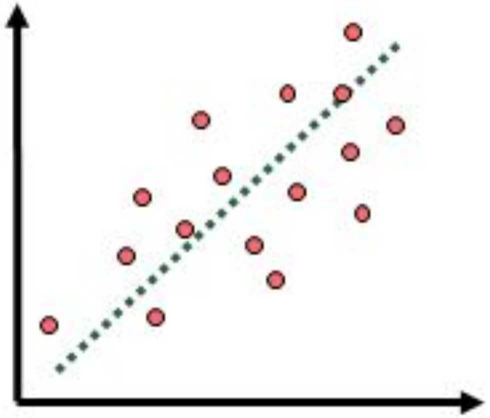

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Notation and definitions:

$\mathcal{X}$ input space

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c(x): $\mathcal{X} \rightarrow \mathcal{Y}$ concept

$\mathcal{C}$ concept class

$\mathcal{H}$ hypothesis class

We receive samples $\mathcal{S} = (x_1, x_2, \ldots, x_m)$ and labels $\mathcal{T} = (y_1,\ y_2,\ \ldots\ y_m)$, where $y_i = c(x_i)$.

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(ii) generalizes to give correct results for new unseen data points drawn from $D_{\mathcal{X}}$.

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Definition: The generalization error (risk) for 0-1 classification $\mathcal{Y}$={0,1} is

$$R(h) = \Pr\{h_S(x) \neq c(x)\} = E_{x \sim D}\left[1_{h_S(x) \neq c(x)}\right]$$

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However, in practice this is NOT computable since we do not known c(x) and D.

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We say a concept class $\mathcal{C}$ is **PAC-learnable** if there exists an algorithm $\mathcal{A}$ and polynomial bound so that given ε > 0 and δ > 0, the following holds for any distribution $D \in \mathbb{D}$ on $\mathcal{X}$, target concept c in $\mathcal{C}$, and sample size m $\geq$ poly(1/ε, 1/δ, n, size(c))

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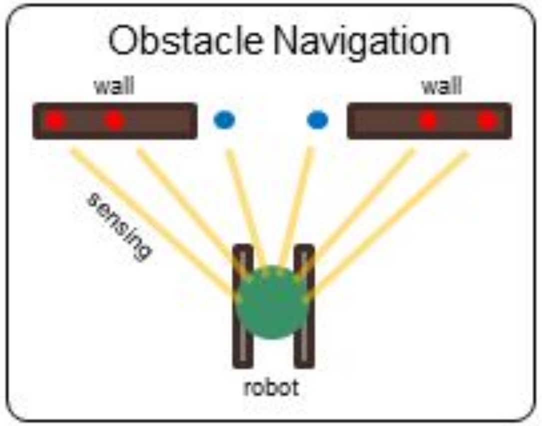
PAC-Learning

Example: Learning intervals on $\mathbb{R}$-line.



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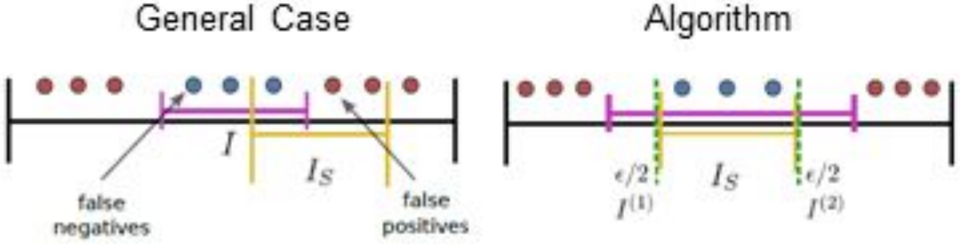

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$\mathscr{A}$: I_S = {smallest interval of in-class}.

$$I_S = [a, b]$$

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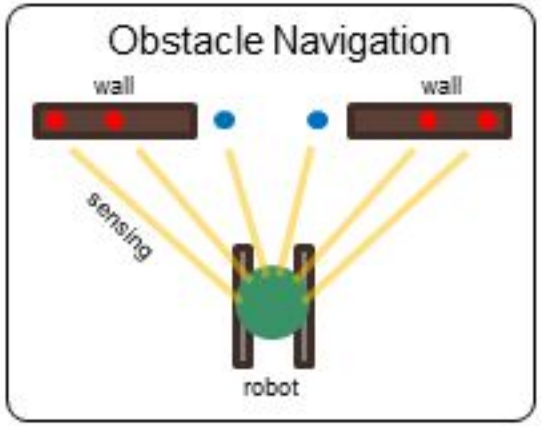

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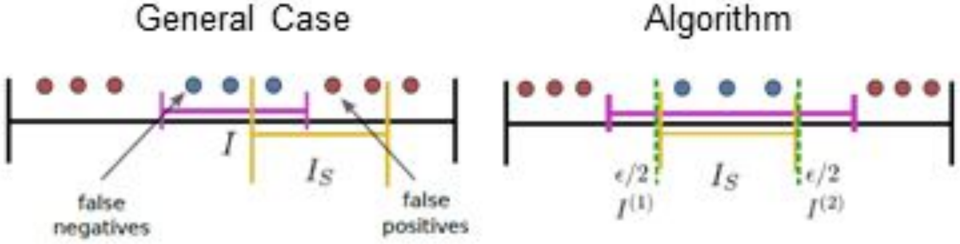

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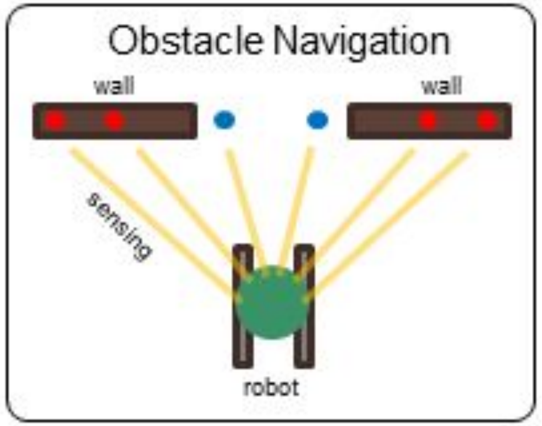

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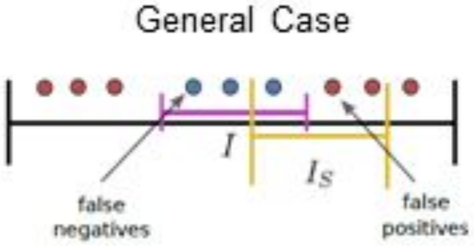
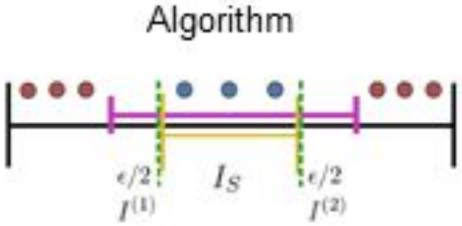

General Case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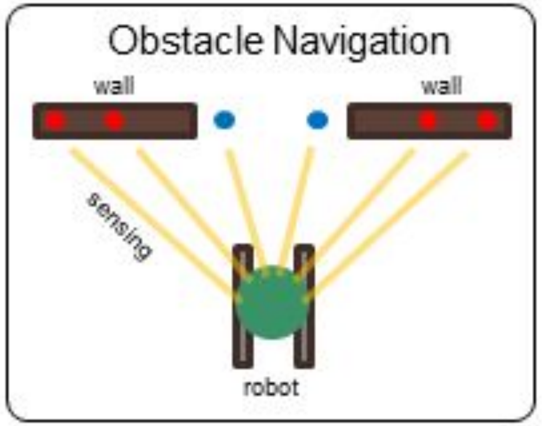
Obstacle Navigation

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Since $I_S \subset I$, we only need to worry about false negatives. This has

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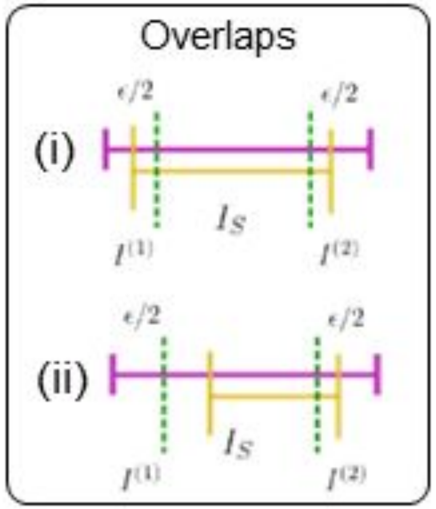
Overlaps

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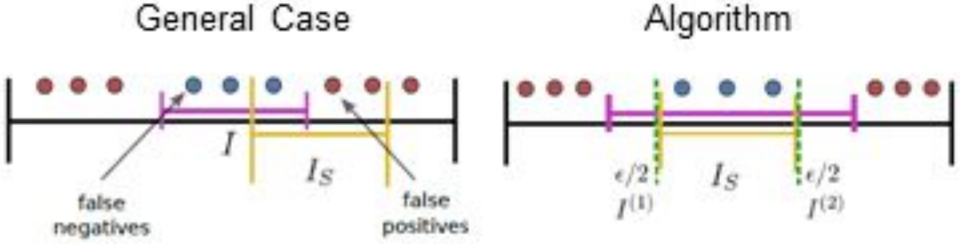

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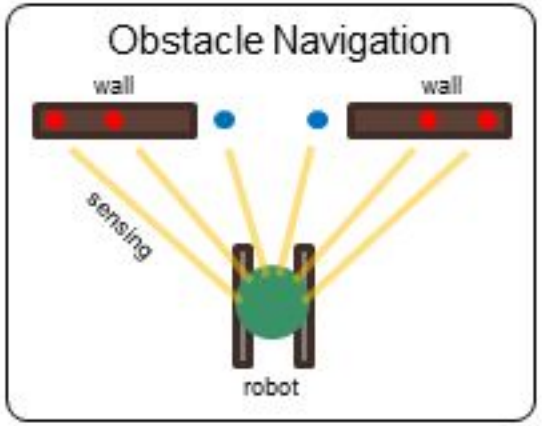

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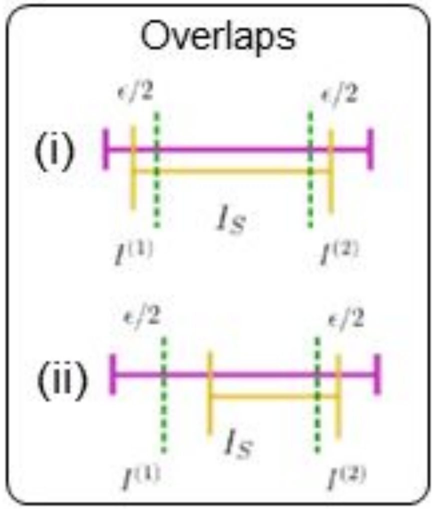
We use that if $\mathcal{A} \Rightarrow \mathcal{B}$ then $\Pr\{\mathcal{A}\} \leq \Pr\{\mathcal{B}\}$ and we use $1 - x \leq \exp[-x]$.

If $I_S \cap I^{(i)} \neq \emptyset, \forall i = 1, 2$ then $R(I_S) \leq \epsilon$. By contrapositive $R(I_S) > \epsilon \Rightarrow \exists i$ s.t $I_S \cap I^{(i)} = \emptyset$.



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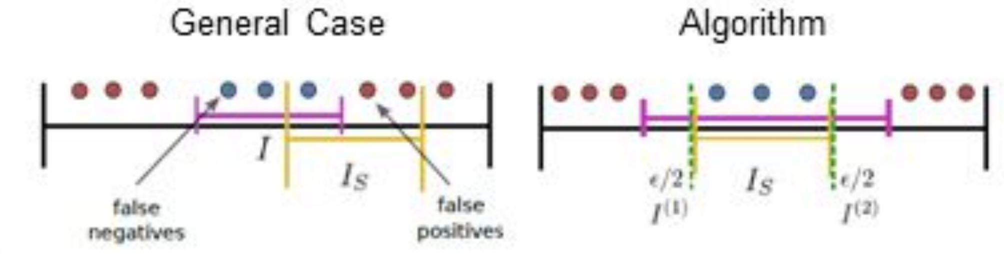

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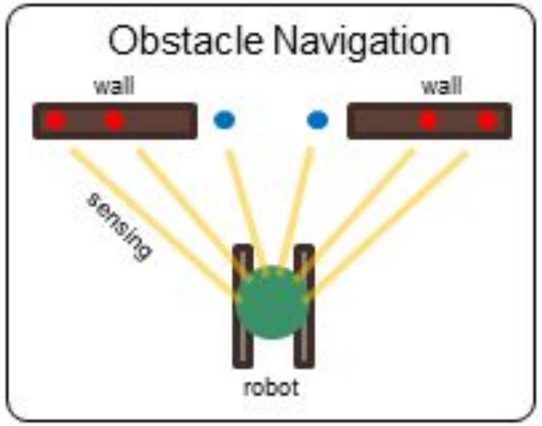

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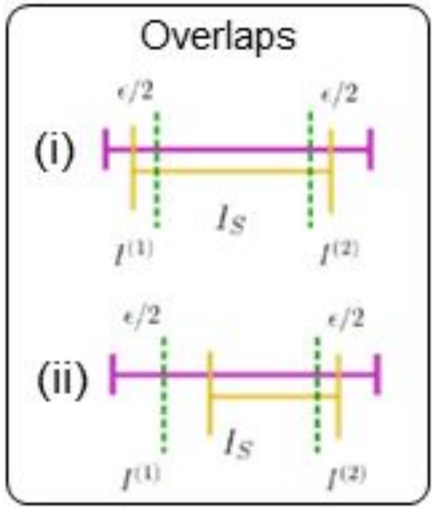
$$R(I_S) = \Pr_{\mathbf{x}\sim D^m}\{x \notin I_S \cap x \in I\} = E_{x\sim D}\left[1_{h_S(x)\neq c(x)}\right].$$

We use that if $\mathcal{A} \Rightarrow \mathcal{B}$ then $\Pr\{\mathcal{A}\} \leq \Pr\{\mathcal{B}\}$ and we use $1 - x \leq \exp[-x]$.

If $I_S \cap I^{(i)} \neq \emptyset, \forall i = 1,2$ then $R(I_S) \leq \epsilon$. By contrapositive $R(I_S) > \epsilon \Rightarrow \exists i$ s.t $I_S \cap I^{(i)} = \emptyset$.

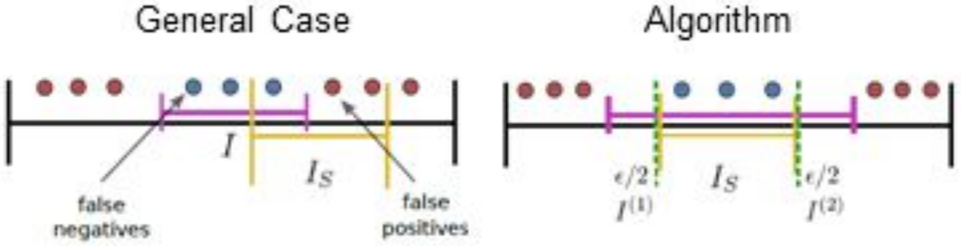
This gives the bound

$$\Pr\{R(I_S) > \epsilon\} \leq \Pr\{\bigcup_{i=1}^{2} I_S \cap I^{(i)} = \emptyset\} \leq \sum_{i=1}^{2} \Pr\{I_S \cap I^{(i)} = \emptyset\} \leq 2\left(1 - \epsilon/2\right)^m \leq 2\exp\left[-\frac{\epsilon m}{2}\right] < \delta$$



PAC-Learning

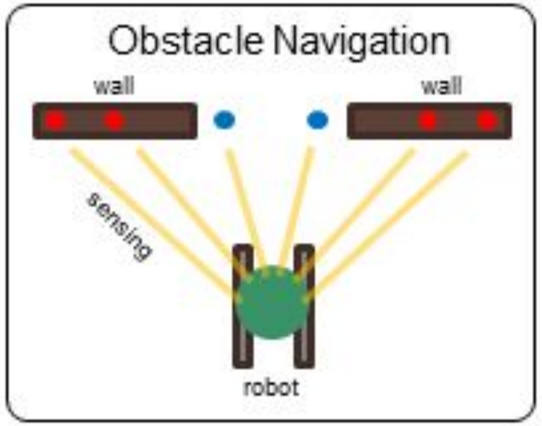
Example: Learning intervals on $\mathbb{R}$-line.



Algorithm:

$\mathcal{A}$: I_S = {smallest interval of in-class}.

$$I_S = [a, b]$$
$$a = \min\{x_i | y_i = 1\},$$
$$b = \max\{x_i | y_i = 1\}.$$



We need to show: Given $\epsilon > 0$, $\delta > 0$ there exists a polynomial bound in samples m with $(\mathcal{S}, \mathcal{T}) = \{(x_i, y_i)\}_{i=1}^m, x_i \in \mathbb{R}, y_i \in \{0,1\}$

$$\Pr_{\mathbf{x} \sim D^m} \{R(I_S) \leq \epsilon\} \geq 1 - \delta.$$

Since $I_S \subset I$, we only need to worry about false negatives. This has

$$R(I_S) = \Pr_{\mathbf{x} \sim D^m} \{x \notin I_S \cap x \in I\} = E_{x \sim D}\left[1_{h_S(x) \neq c(x)}\right].$$

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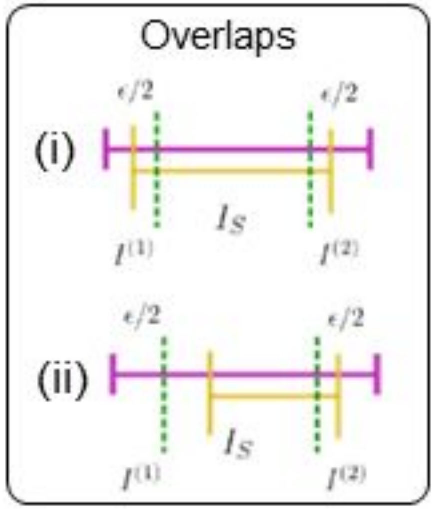
$$\Rightarrow m > \frac{2}{\epsilon} \ln\left(\frac{2}{\delta}\right). \blacksquare$$



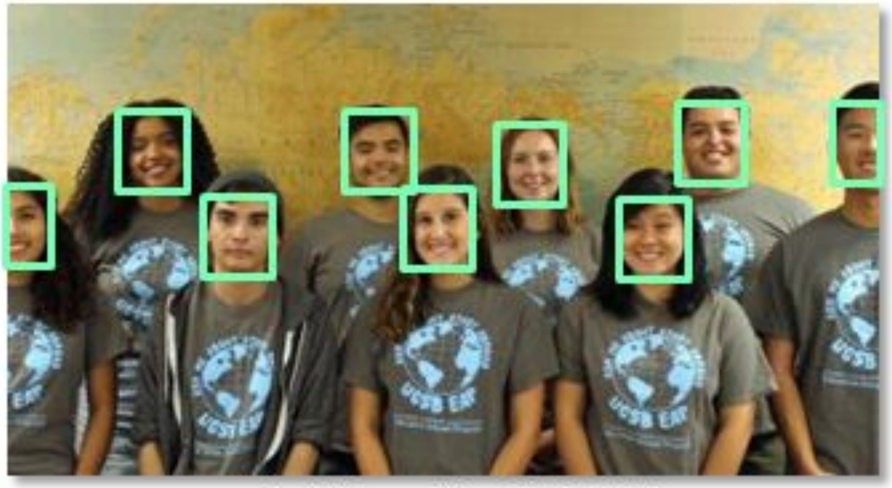
PAC-Learning

Example: Learning axis-aligned rectangles.

Building Identification

Google Maps: UCSB South Hall

Picture Annotation



Facial Recognition: UCSB EAP

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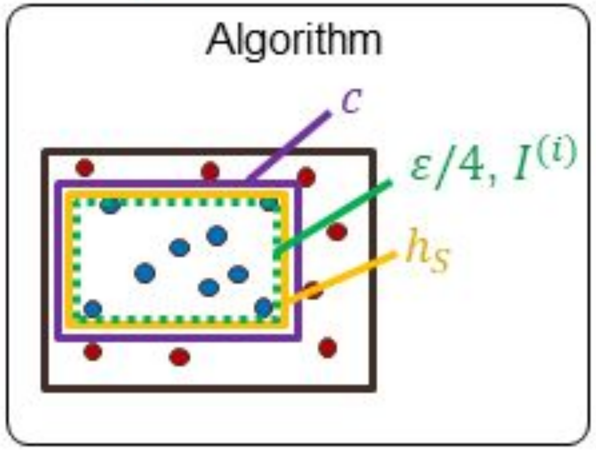

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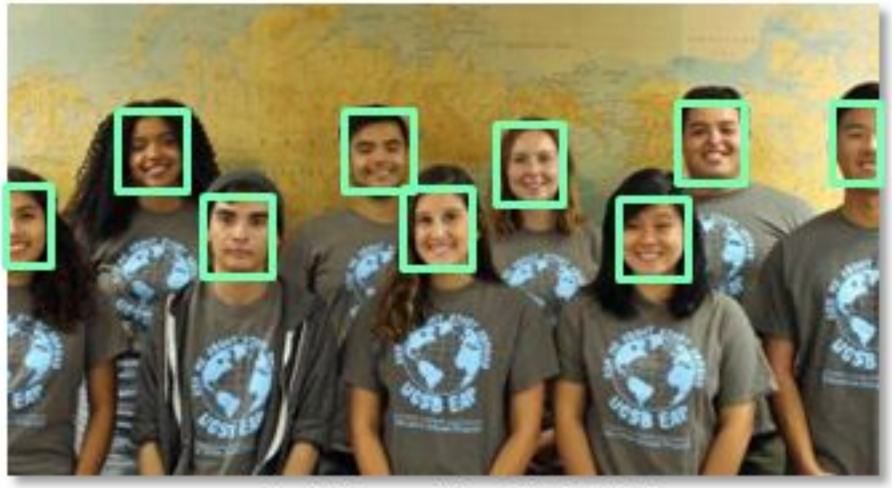

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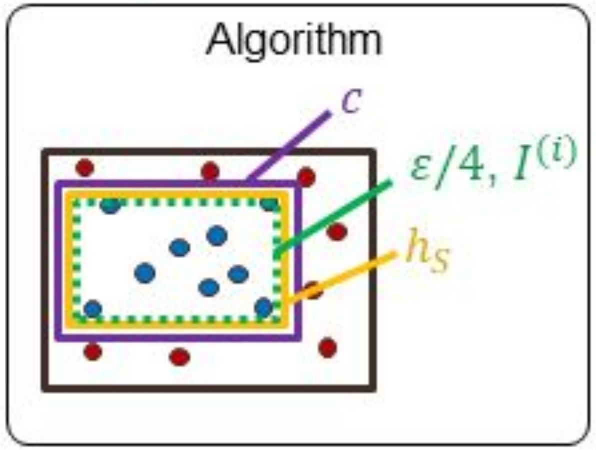

We need to show

$$\Pr\{R(h_S) \leq \epsilon\} \geq 1 - \delta$$

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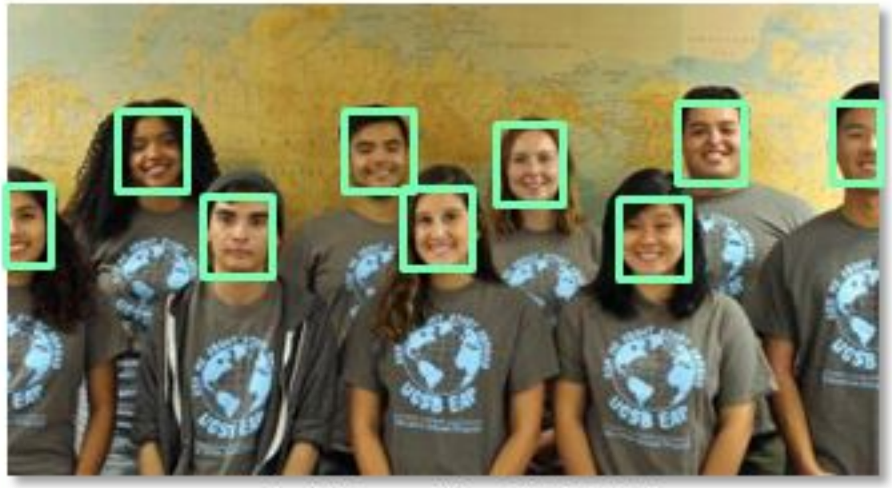

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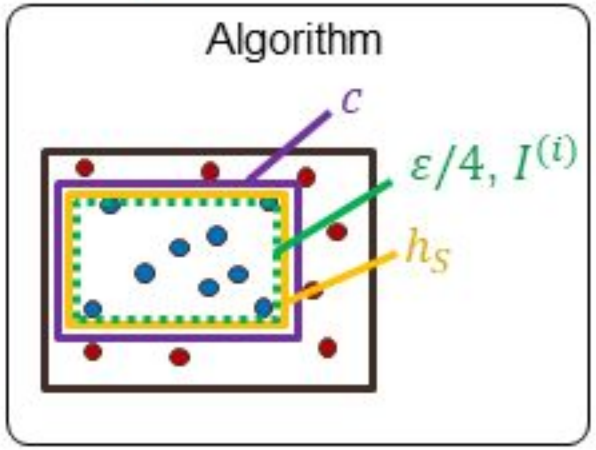

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$$R(h_S) > \epsilon \;\Rightarrow \exists i \text{ s.t. } h_S \cap I^{(i)} = \emptyset$$

Building Identification

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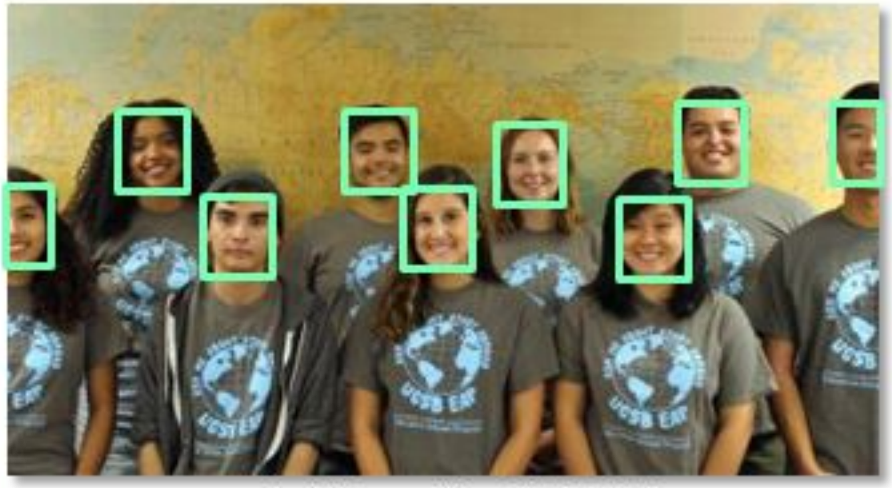

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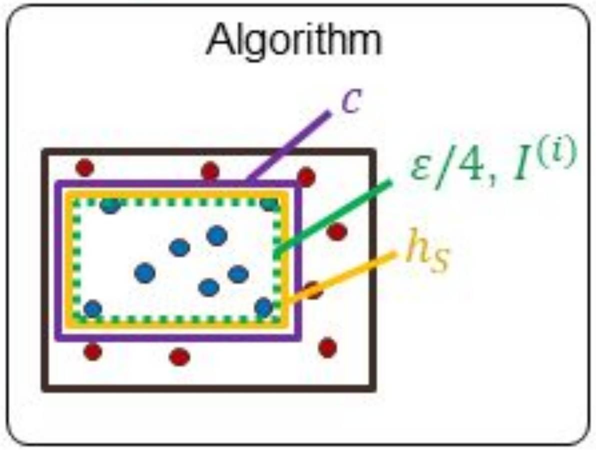

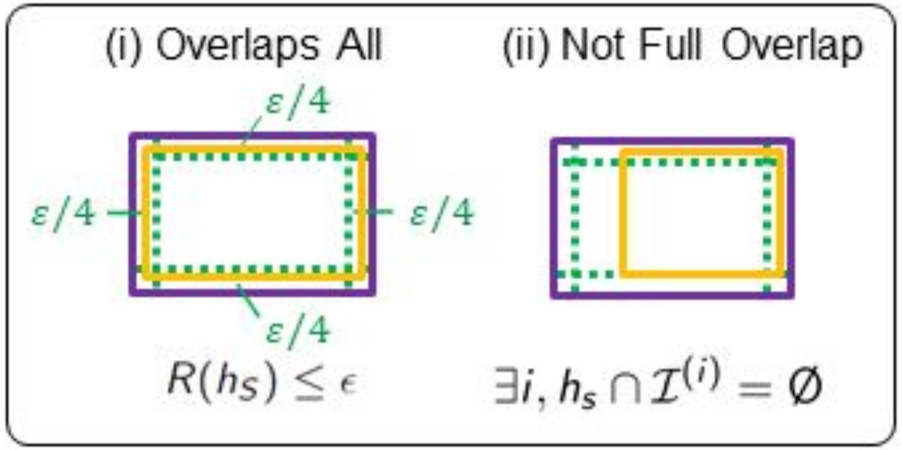

$R(h_S) \leq \epsilon$

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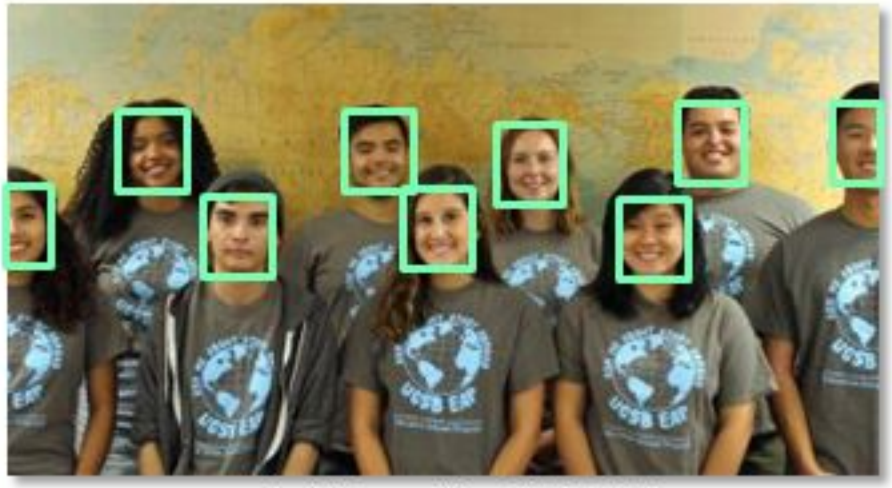

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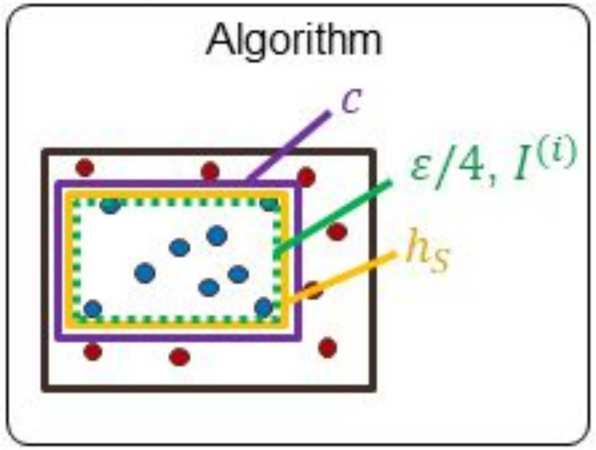

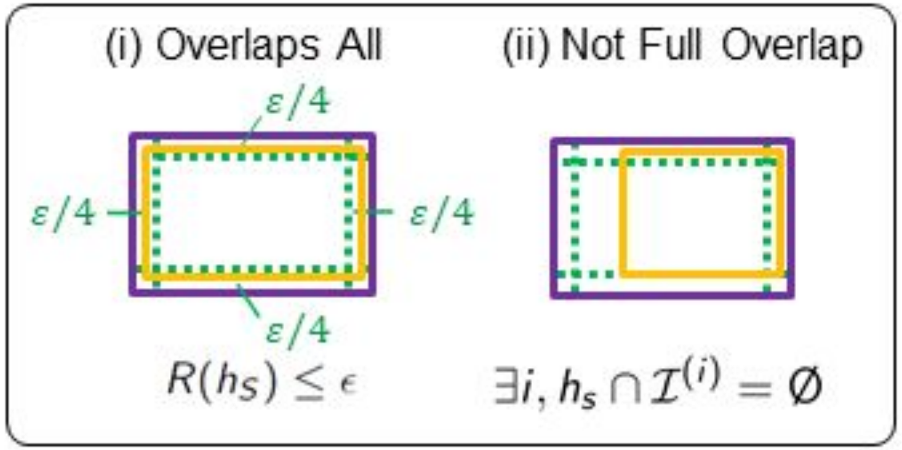

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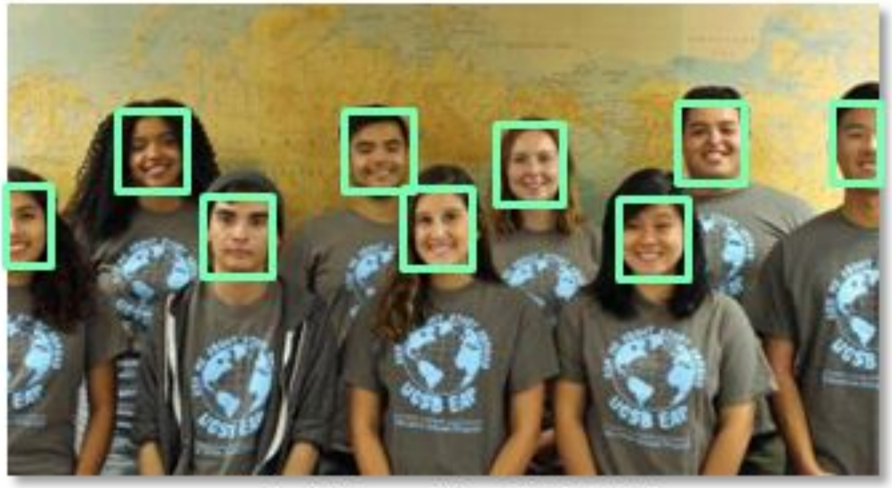

Facial Recognition: UCSB EAP

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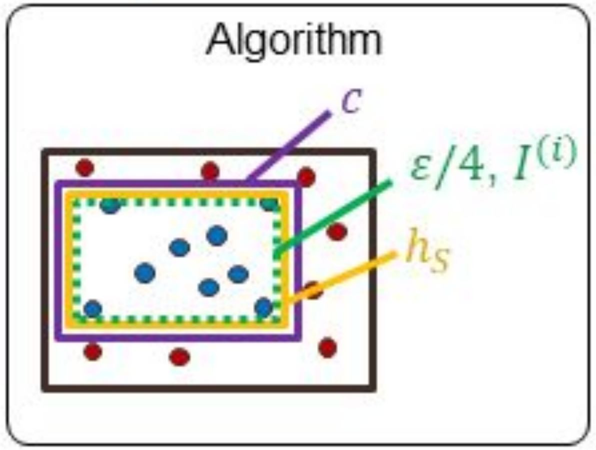

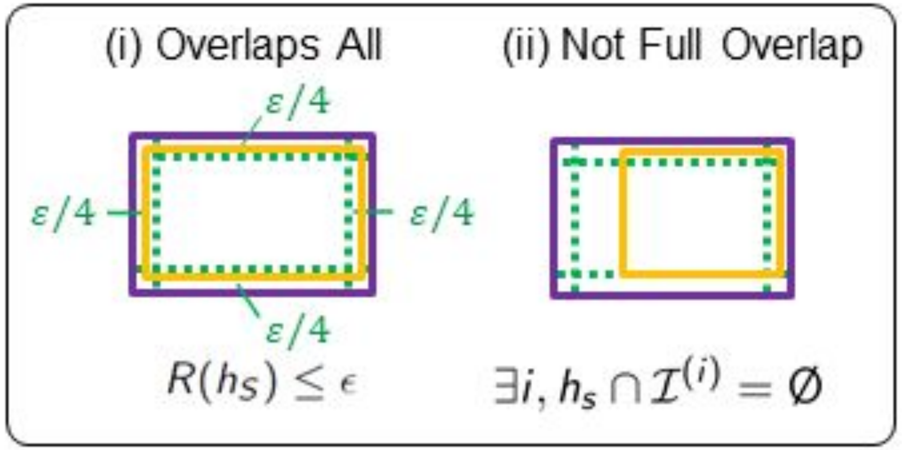

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Building Identification

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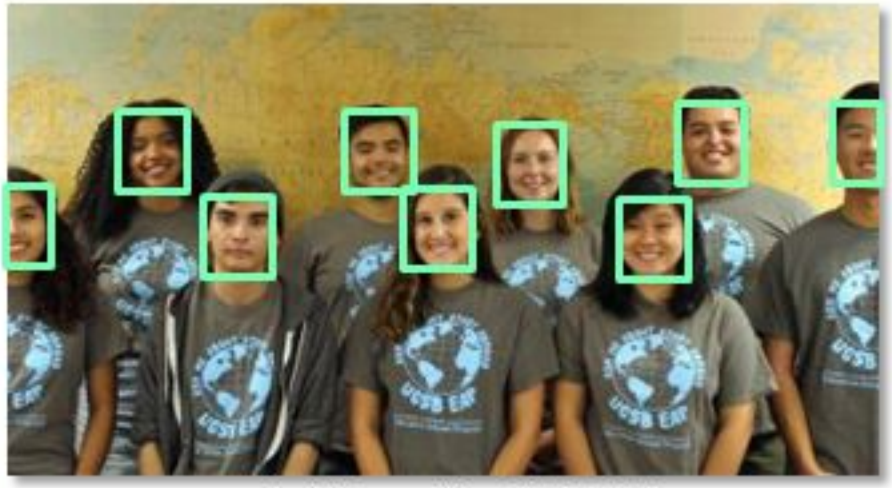

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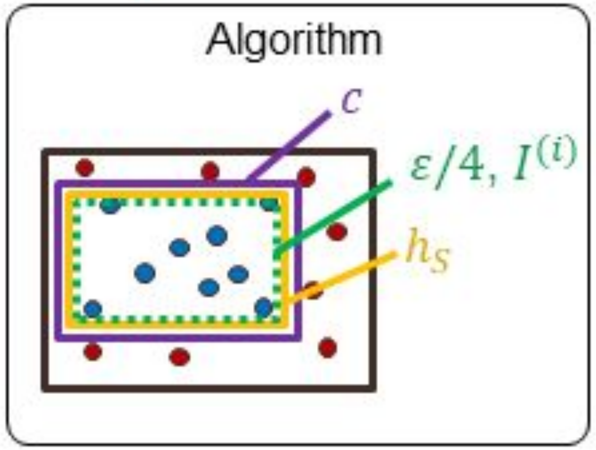

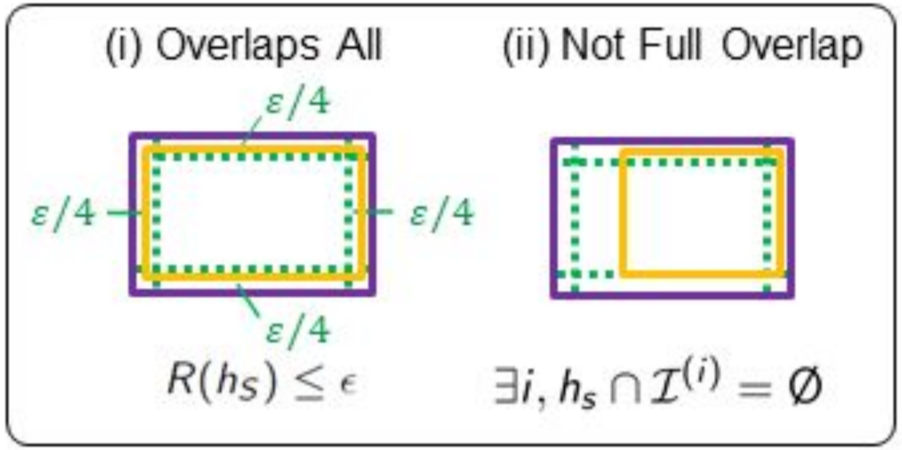

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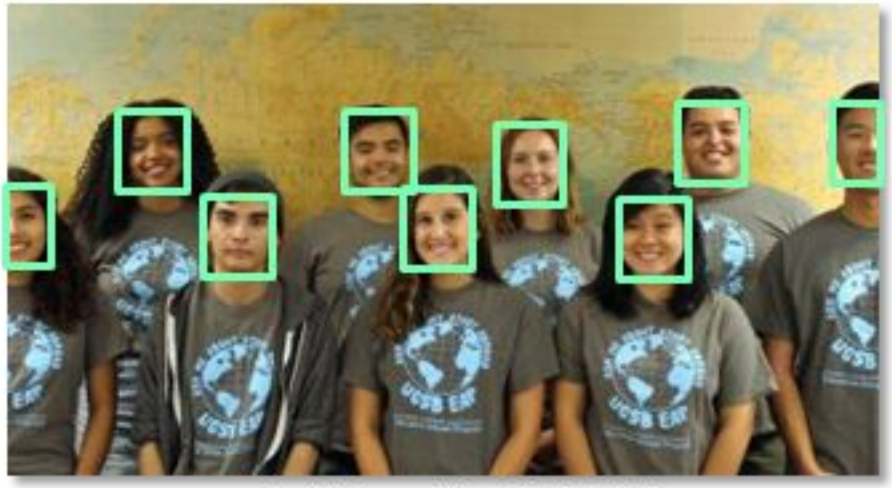

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Bound on samples m

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PAC-Learning

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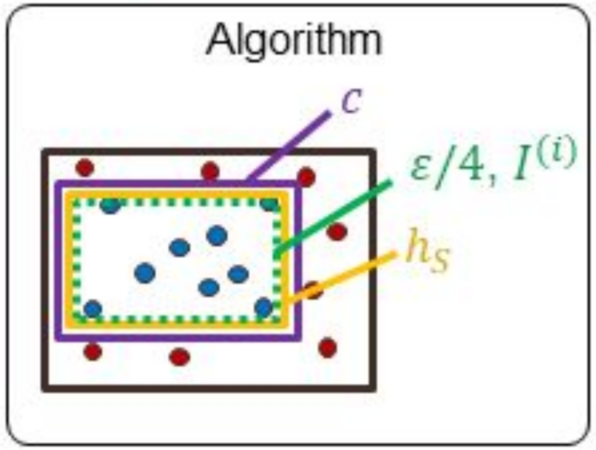

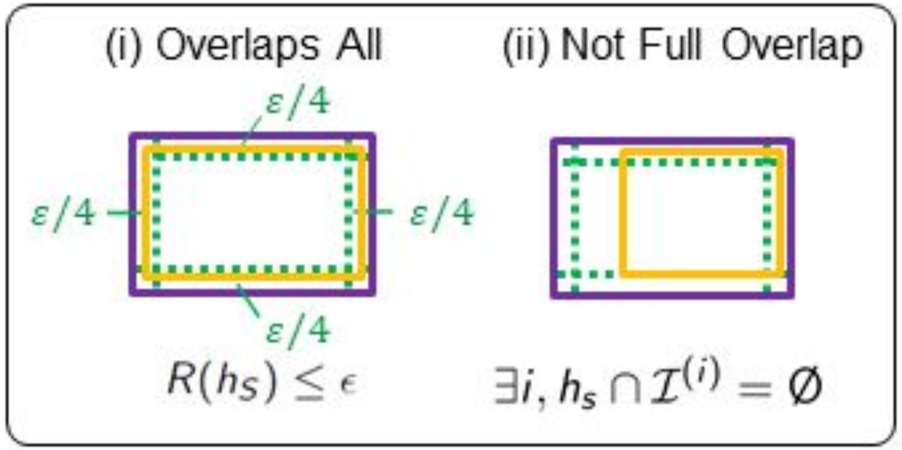

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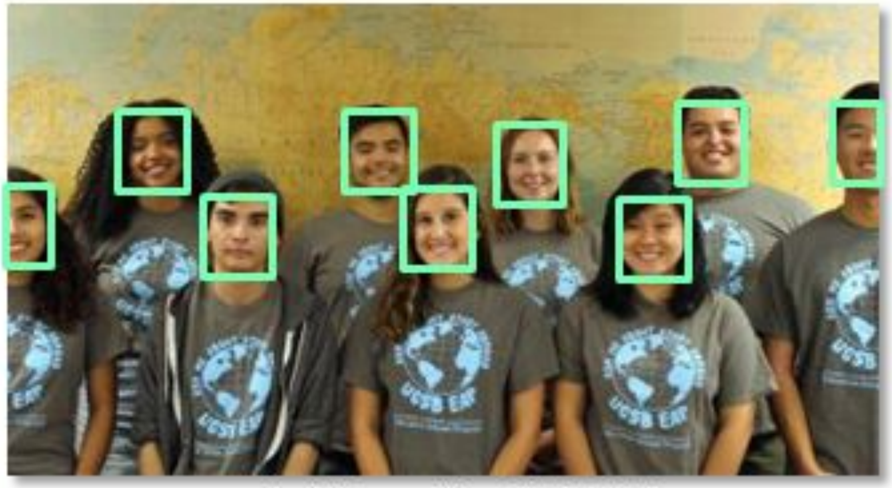

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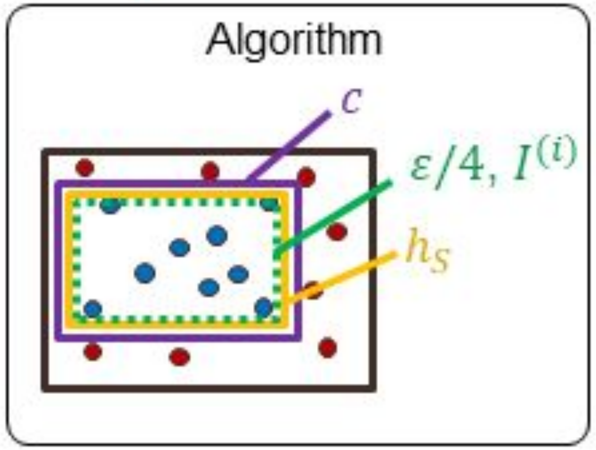

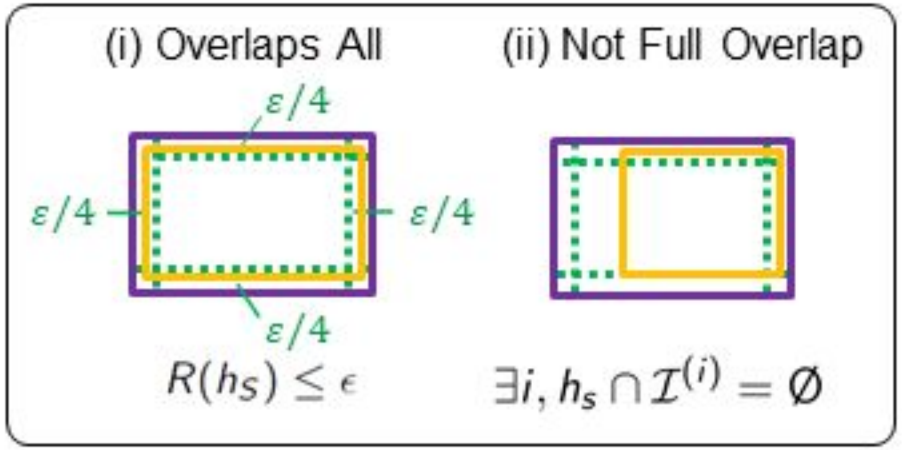

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Bound on risk R

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Building Identification

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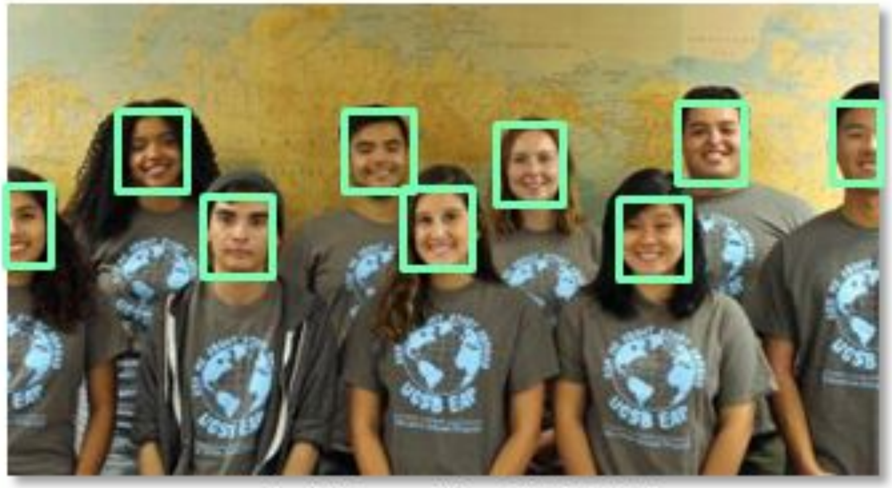

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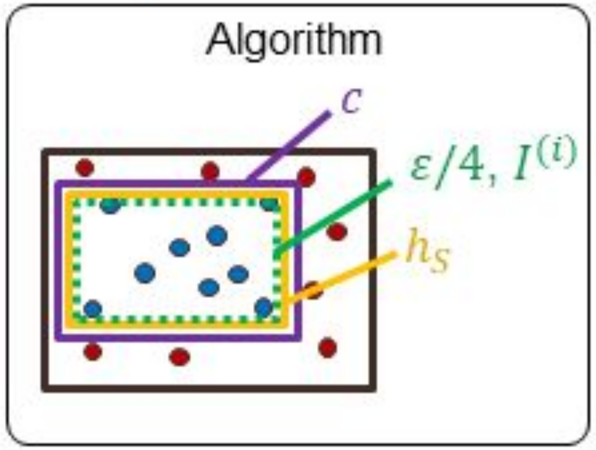

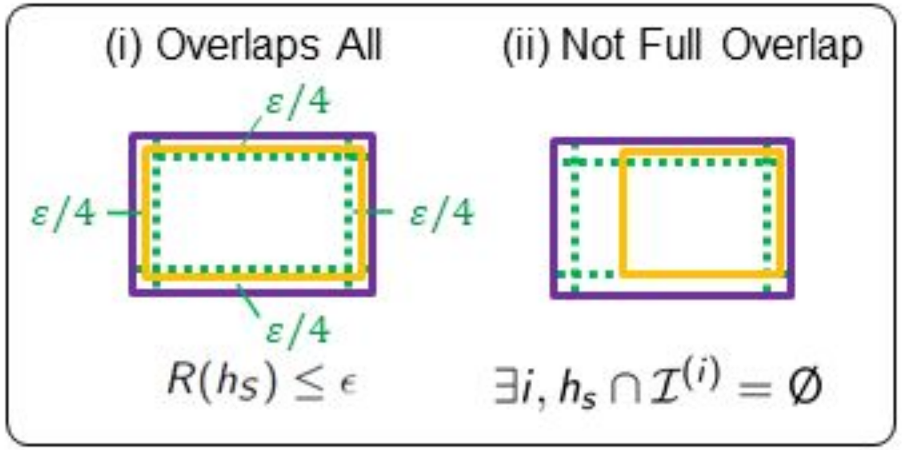

(i) Overlaps All

$R(h_S) \le \epsilon$

(ii) Not Full Overlap

$\exists i,\ h_s \cap \mathcal{I}^{(i)} = \emptyset$

Building Identification

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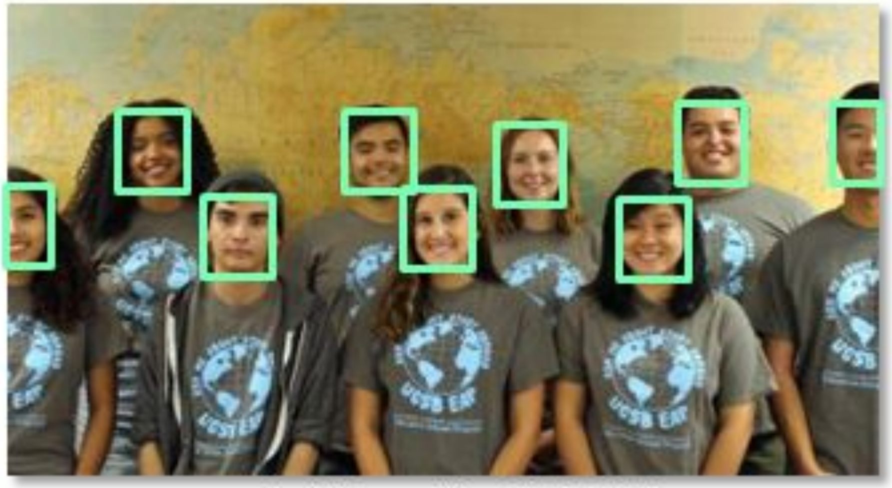

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Data Sampling Complexity

Guarantees on Sampling Complexity

How many samples do we need to guarantee a given level of precision ϵ, δ in PAC-learning?

What is bound M so for $m \geq M$ we have $\Pr\{R(h_S) \leq \epsilon\} \geq 1 - \delta$?

Empirical Generalization Error

$$\hat{R}(h) = \frac{1}{m} \sum_{i=1}^{m} 1_{h_S(x_i) \neq c(x_i)}$$

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Two important cases:

(i) consistent case: $\mathcal{C} \subset \mathcal{H}$, hypotheses include all concepts.

(ii) inconsistent case: $\mathcal{C} \not\subset \mathcal{H}$, hypotheses can not capture all concepts.

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Distinguish also case of finite vs infinite hypothesis spaces $\mathcal{H}$ and concept spaces $\mathcal{C}$.

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Distinguish also case of finite vs infinite hypothesis spaces $\mathcal{H}$ and concept spaces $\mathcal{C}$.

Theorem: Consistent-Finite Hypothesis Spaces $\mathcal{H}$. Let $\mathcal{A}$ be any learning algorithm that has zero Empirical Generalization Error $\hat{R}(h_S) = 0$ then PAC-learning bound $\Pr\{R(h_S) \leq \epsilon\} \geq 1 - \delta$ is guaranteed to hold for m samples satisfying

$$m \geq \frac{1}{\epsilon}\left(\log |H| + \log \frac{1}{\delta}\right)$$

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Data Sampling Complexity

Finite Consistent-Case: Guarantees on Sampling Complexity

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Proof: Let $\mathscr{A}$ be any algorithm that returns for m samples S a hypothesis h_S s.t. $\hat{R}(h_S) = 0.$

Data Sampling Complexity

Finite Consistent-Case: Guarantees on Sampling Complexity

Theorem: Consistent-Finite Hypothesis Spaces $\mathscr{H}$. Let $\mathscr{A}$ be any learning algorithm that has zero empirical generalization error $\hat{R}(h_S) = 0$ then PAC-learning bound $\Pr\{R(h_S) \leq \epsilon\} \geq 1 - \delta$ is guaranteed to hold for m samples satisfying

$$m \geq \frac{1}{\epsilon}\left(\log|H| + \log\frac{1}{\delta}\right)$$

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$$\begin{aligned}\Pr\{A \wedge B \wedge C\} &= \Pr\{A \wedge B | C\}\Pr\{C\} \\ &\leq \Pr\{A \wedge B\}\end{aligned}$$

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Finite-Consistent Case: Guarantees on Sampling Complexity

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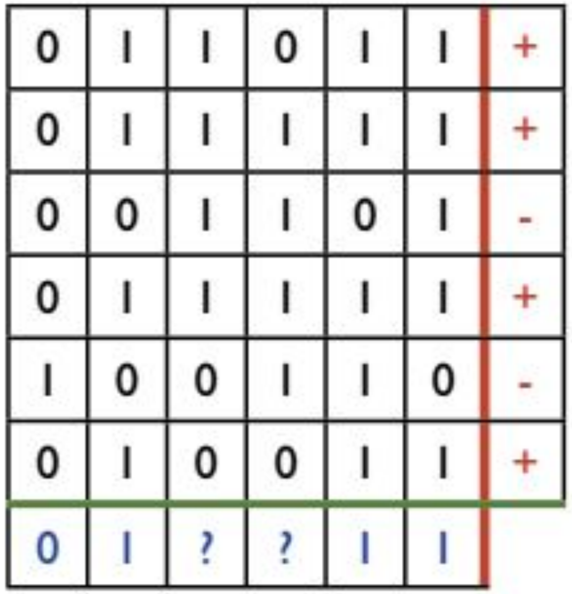
- However, **consistency** $\mathcal{C} \subset \mathcal{H}$ requires **"big enough" hypothesis space** $\mathcal{H}$ to capture target concepts.

Data Sampling Complexity

Example: Boolean Conjunctions.

Let z_i be Boolean variable, a conjunction is: $c = \overline{z_1} \wedge z_2 \wedge \; z_5 \wedge z_6$.



0	1	1	0	1	1	+
0	1	1	1	1	1	+
0	0	1	1	0	1	-
0	1	1	1	1	1	+
1	0	0	1	1	0	-
0	1	0	0	1	1	+
0	1	?	?	1	1	

Mohri 2012

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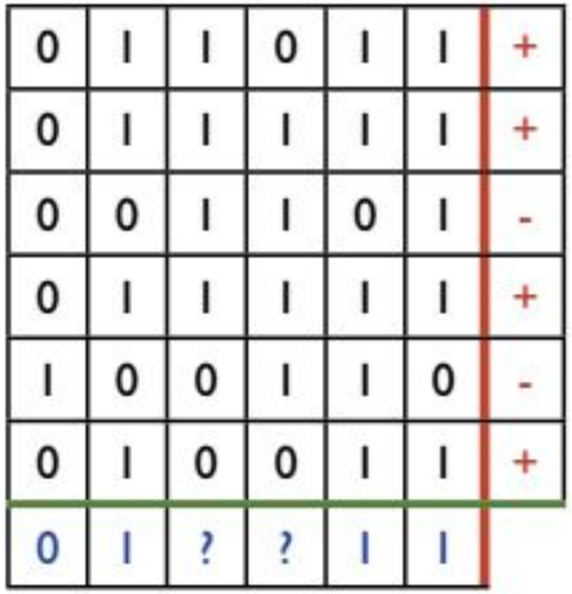

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Mohri 2012

Example 2:

$$C = z_1 \wedge \bar{z}_2 \wedge z_3$$

$z_1 \sim$ "it is raining"
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0	0	I	I	0	I	-
0	I	I	I	I	I	+
I	0	0	I	I	0	-
0	I	0	0	I	I	+
0	I	?	?	I	I	

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Bound on number samples:

$m \geq 630$

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Mohri 2012

The concept class $|\mathcal{C}_n| = 3^n$, since in n-conjunction either z_i, $\overline{z_i}$, or ϕ.

Note could learn directly with as few as 2n examples if special ones chosen.

Let $\mathcal{H} = \mathcal{C}_n$ then we have consistent-finite hypothesis space and $\hat{R}(h_S) = 0$.
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Example 2:

$$C = z_1 \wedge \bar{z}_2 \wedge z_3$$

$z_1 \sim$ "it is raining"
$z_2 \sim$ "have umbrella"
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Confidence desired:

$\epsilon = 0.01 \to 99\%$
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(larger than direct testing 2n)

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Learning completely generic functions is just too hard to do efficiently (too many possibilities).

Agnostic PAC-Learning

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Stochastic vs Deterministic Learning: Above applies also when label **y** for feature vector **x** is not unique, as in many real-world data sets. Uncertainty captured by D ~ $\mathcal{X}$ x $\mathcal{Y}$, allowing for a type of stochastic learning. **Goal:** Find best assignment y = h(x) minimizing generalization error.

Generalization Bounds

Finite-Inconsistent Case: Guarantees on Sampling Complexity

Theorem: Inconsistent-Finite Hypothesis Spaces $\mathcal{H}$. Let $\mathcal{A}$ be any learning algorithm that has empirical generalization error $\hat{R}(h_S)$ then for any h ∈ $\mathcal{H}$ we have

$$R(h) \leq \widehat{R}(h) + \sqrt{\frac{\log|H| + \log\frac{2}{\delta}}{2m}} \quad \blacksquare$$

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Note, only $\mathrm{m}^{-1/2}$ scaling in the bound (compare to the finite-consistent case ~ m^{-1}).

Probability Theory and Inequalities

Concentration Inequalities

Lemma: Markov Inequality $\Pr[X \geq \epsilon] = \Pr[e^{tX} \geq e^{t\epsilon}] \leq e^{-t\epsilon}\,\mathrm{E}[e^{tX}]$ for $t \geq 0$.

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Proof: We have that $e^{tx} \leq \frac{b-x}{b-a}e^{ta} + \frac{x-a}{b-a}e^{tb}$ using $a \leq x \leq b$, $x \to e^{tx}$ is a convex function.

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Probability Theory and Inequalities

Concentration Inequalities

Lemma: Markov Inequality $\Pr[X \geq \epsilon] = \Pr[e^{tX} \geq e^{t\epsilon}] \leq e^{-t\epsilon}\,\mathrm{E}[e^{tX}]$ for $t \geq 0$.

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Probability Theory and Inequalities

Concentration Inequalities

Lemma: (Hoeffding's Inequality) Let $X_1, X_2, \ldots, X_m$ be random variables with $a_i \leq X_i \leq b_i$, $b_i > a_i$ and $S_m = \sum_{i=1}^{m} X_i$ then we have the bounds

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Probability Theory and Inequalities

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$$\psi(t) = \frac{-8t\epsilon + t^2 Q}{8}$$

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Probability Theory and Inequalities

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Similarly, we obtain the other case using $\tilde{Z}_m = -Z_m$. ■

Generalization Bounds

Finite-Inconsistent Case: Guarantees on Sampling Complexity

Lemma: Let samples S={(x_1,y_1),(x_2,y_2),…,(x_m,y_m)} be chosen i.i.d. on {0,1} from D ~ $\mathcal{X}$ x $\mathcal{Y}$ then

$$\Pr_{S \sim D^m}\left[|\widehat{R}(h) - R(h)| \geq \epsilon\right] \leq 2\exp(-2m\epsilon^2)$$

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Proof:

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By Hoeffding's Inequality

$$\Pr\{|\hat{R}(h) - R(h)| \geq \epsilon\} \leq 2e^{-2\epsilon^2/\sum_{i=1}^{m}(b_i-a_i)^2} = 2e^{\frac{-2\epsilon^2 m^2}{m}} = 2\exp\left(-2\epsilon^2 m\right) \blacksquare$$

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Theorem: Inconsistent-Finite Hypothesis Spaces $\mathcal{H}$. Let $\mathcal{A}$ be any learning algorithm that has empirical generalization error $\hat{R}(h_S)$ then for any h $\in \mathcal{H}$ we have with probability at least $1-\delta$

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Proof:

$$\Pr\{h \in \mathcal{H}, |\hat{R}(h) - R(h)| > \epsilon\} = \Pr\{h_1 \in \mathcal{H} \wedge |\hat{R}(h_1) - R(h_1)| > \epsilon \vee \cdots \vee h_{|\mathcal{H}|} \in \mathcal{H} \wedge |\hat{R}(h_{|\mathcal{H}|}) - R(h_{|\mathcal{H}|})| > \epsilon\}$$

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Finite-Inconsistent Case: Guarantees on Sampling Complexity

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$$R(h) \leq \widehat{R}(h) + \sqrt{\frac{\log |H| + \log \frac{2}{\delta}}{2m}}$$

Proof:

$$\begin{aligned}\Pr\{h \in \mathcal{H}, |\hat{R}(h) - R(h)| > \epsilon\} = \; & \Pr\{h_1 \in \mathcal{H} \wedge |\hat{R}(h_1) - R(h_1)| > \epsilon \vee \cdots \vee h_{|\mathcal{H}|} \in \mathcal{H} \wedge |\hat{R}(h_{|\mathcal{H}|}) - R(h_{|\mathcal{H}|})| > \epsilon\} \\ & \leq \sum_{i=1}^{|\mathcal{H}|} \Pr\{h_i \in \mathcal{H} \wedge |\hat{R}(h_i) - R(h_i)| > \epsilon\} \leq |\mathcal{H}| 2 \exp(-2m\epsilon^2) \leq \delta\end{aligned}$$

Generalization Bounds

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$$\Rightarrow \log(|\mathcal{H}|) - 2m\epsilon^2 \leq \log\left(\frac{\delta}{2}\right) \Rightarrow 2m\epsilon^2 \geq \log(|\mathcal{H}|) + \log\left(\frac{2}{\delta}\right)$$

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& \qquad\qquad\qquad\qquad \Rightarrow \quad m \geq \frac{1}{2\epsilon^2}\left(\log(|\mathcal{H}|) + \log\left(\frac{2}{\delta}\right)\right)
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Generalization Bounds

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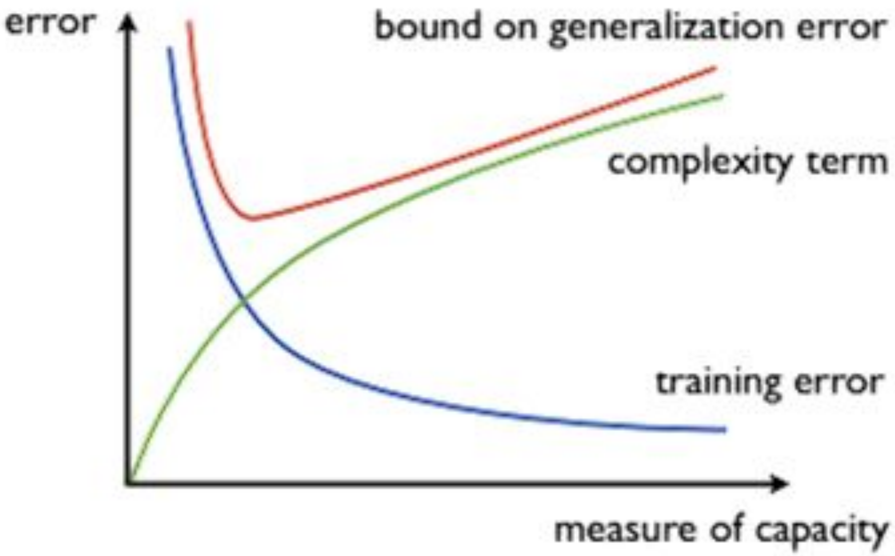
$$\Rightarrow \epsilon \geq \sqrt{\frac{\log\left(|\mathcal{H}|\right) + \log\left(\frac{2}{\delta}\right)}{2m}}$$

■

Generalization Behaviors

Generalization Error and Model Capacity



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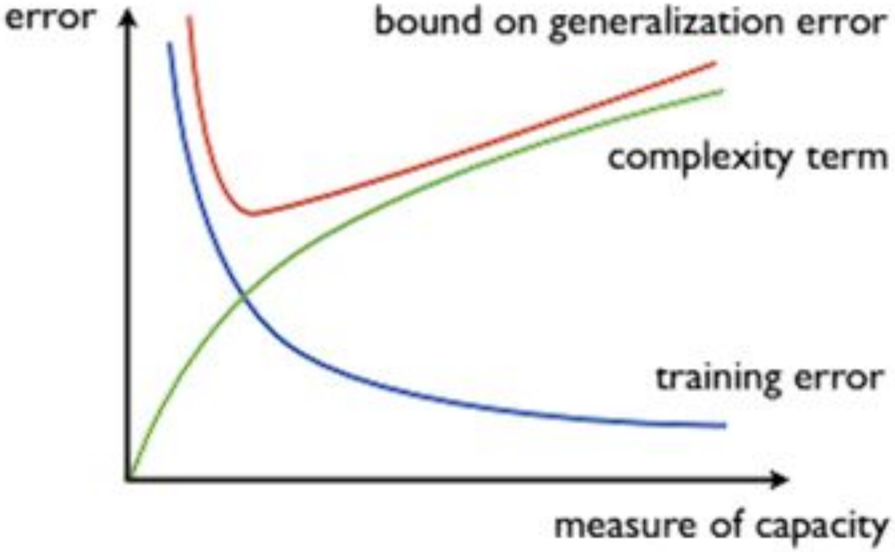
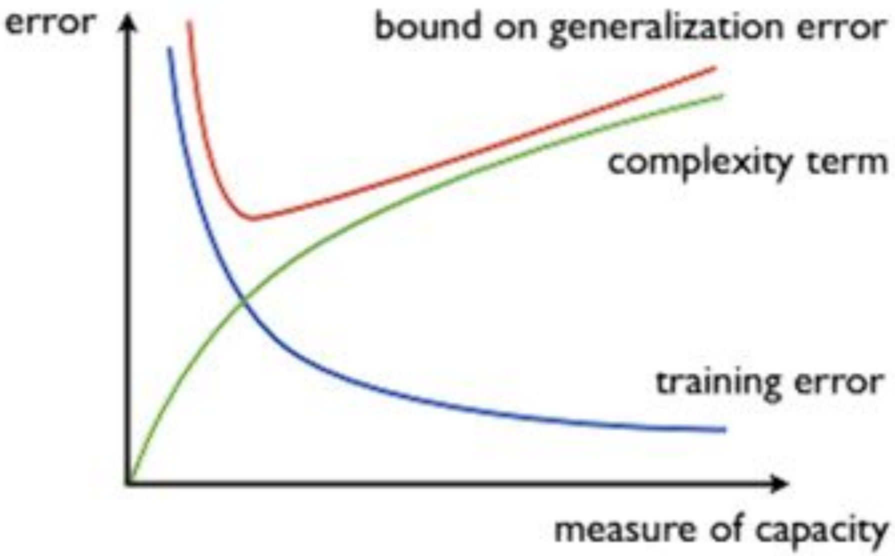


- **Larger model capacity** often allows for **smaller training error** (model capacity ~ $|\mathcal{H}|$).

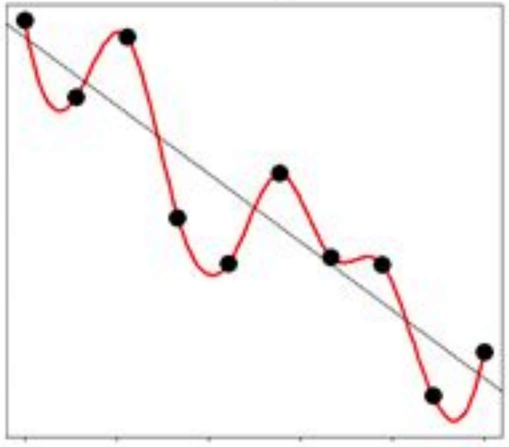
- **Complexity of** $\mathcal{H}$ tends to hinder generalization to new inputs.

Generalization Behaviors

Generalization Error and Model Capacity



Overfitting data



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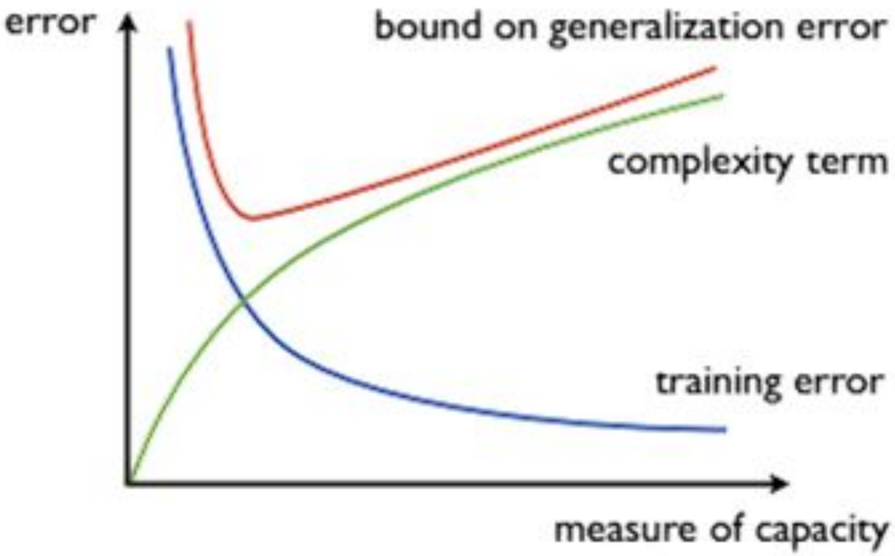

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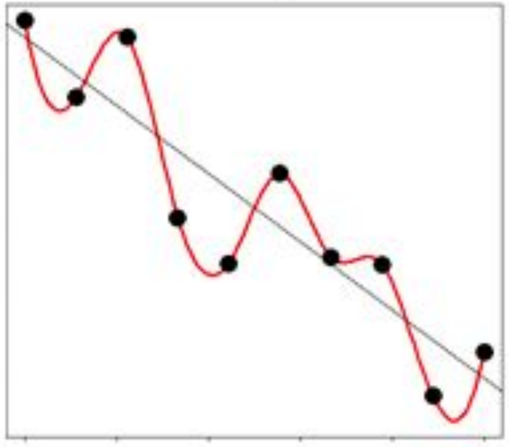

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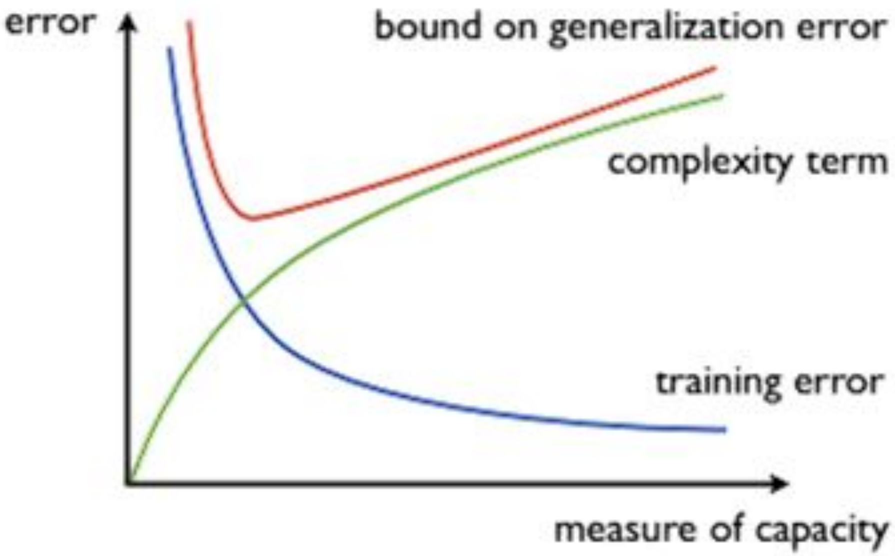
Generalization Error and Model Capacity



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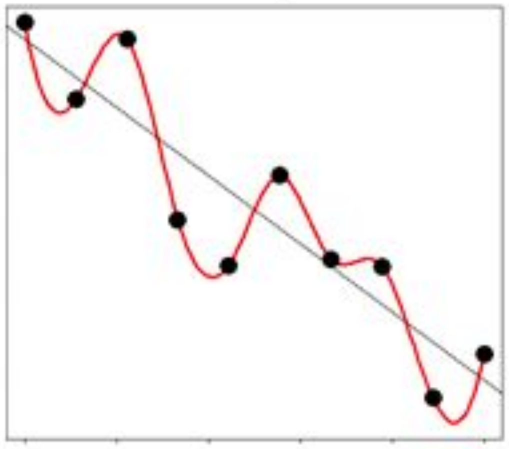

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- **Complexity of** $\mathcal{H}$ tends to hinder generalization to new inputs.
- **Smallest generalization error** arises intermediate trading-off in model complexity and training error.
- **Central challenge in machine learning** is to find appropriate hypothesis classes for given learning tasks.

Minimax Rates and PAC-Learning

Minimax Rate

$$\mathcal{V}_m(\mathcal{C}) = \inf_{h_S = \mathcal{A}(\cdot)} \sup_{D_X, c \in \mathcal{C}} E_{S:|S|=m}\left[R(h_S)\right]$$

$\mathcal{X}$ input space, $\mathcal{Y}$ output space, c(x): $\mathcal{X} \rightarrow \mathcal{Y}$ concept
$\mathcal{C}$ concept class, $\mathcal{H}$ hypothesis class.

Minimax Rates and PAC-Learning

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PAC-learning Classification:

$$\mathcal{V}_m^{PAC}(\mathcal{C}) = \inf_{h_S = \mathcal{A}(\cdot)} \sup_{D_X, c \in \mathcal{C}} E_{S:|S|=m}\left[\Pr_{x \sim D}\left\{h_S(x) \neq c(x)\right\}\right]$$

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A concept class $\mathcal{C}$ is PAC-learnable if $\mathcal{V}_m^{PAC}(\mathcal{C}) \to 0$.

More precisely, given $\epsilon > 0$, $\exists M = \mathrm{poly}(1/\epsilon)$ such that $m \geq M$, we have

$\mathcal{V}_m^{PAC}(\mathcal{C}) \leq \epsilon$.

Minimax Rates and PAC-Learning

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Theorem (PAC Learning ←→ Minimax): For a concept class $\mathcal{C}$ the minimax rate converges to zero with polynomial sampling complexity if and only if the concept class $\mathcal{C}$ is PAC-learnable.

Minimax Rates and PAC-Learning

Minimax Rate and PAC-Learning Classification $\mathcal{V}^{PAC}(\mathcal{C}) = \inf_{\hat{\mathcal{A}}} \sup_{\mathcal{D}_X, c \in \mathcal{C}} E_{S:|S|=m}\left[R(h_S = \mathcal{A}(S)\right]$

Theorem (PAC Learning ←→ Minimax):

Given $\epsilon > 0$, $\mathcal{V}_m^{PAC}(\mathcal{C}) \leq \epsilon$ with $m \geq \mathrm{poly}(1/\epsilon)$ holds if and only if there is an algorithm $\mathcal{A}$ so that given $\epsilon > 0$, $\delta > 0$, $\Pr_{S \sim D^m}\left\{R(h_S) \leq \epsilon\right\} \geq 1 - \delta$ for $m \geq \mathrm{poly}(1/\epsilon, 1/\delta)$ holds.

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Proof: $(i) \Rightarrow (ii)$ follows readily.

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Given (ii) we have $\exists \tilde{\tilde{\mathcal{A}}}$ s.t. given $\epsilon > 0$, $\delta = \epsilon/2$, $\exists M = \text{poly}\left(\frac{1}{\epsilon}, \frac{1}{\epsilon}\right)$ s.t. for $D_X \in \mathfrak{D}$, $c \in C$,

$$\Pr_{S \sim \mathcal{D}^m}\{R(\tilde{\tilde{\mathcal{A}}}(S)) \leq \epsilon\} \geq 1 - \delta \Rightarrow \Pr_{S \sim \mathcal{D}^m}\{R(\tilde{\tilde{\mathcal{A}}}(S)) > \epsilon\} < \delta,\ m \geq M.$$

Minimax Rates and PAC-Learning

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We obtain the bound

$$E_{S:|S|=m}\left[R(\tilde{\tilde{\mathcal{A}}}(S))\right] \leq \Pr_{S\sim\mathcal{D}^m}\{R(\tilde{\tilde{\mathcal{A}}}(S) \leq \epsilon\} \cdot \epsilon + \Pr_{X\sim\mathcal{D}}\{R(\tilde{\tilde{\mathcal{A}}}(S) > \epsilon\} \cdot 1 \leq \epsilon + \delta \leq \epsilon + \frac{1}{2}\epsilon = \frac{3}{2}\epsilon = \tilde{\epsilon}.$$

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$$\Rightarrow \begin{cases} \mathcal{V}^{PAC}(\mathcal{C}) \leq \tilde{\epsilon} \\ m \geq \text{poly}(1/\tilde{\epsilon}) \end{cases}$$

$\Rightarrow \mathcal{V}_m^{PAC} \to 0$, as $m \to \infty$. ■

Minimax Rates and Learning Tasks

PAC-Learning Classification

$$\mathcal{V}_m^{PAC}(\mathcal{C}) = \inf_{h_S = \mathcal{A}(\cdot)} \sup_{D_X, c \in \mathcal{C}} E_{S:|S|=m}\left[\Pr_{x \sim D}\left\{h_S(x) \neq c(x)\right\}\right]$$

Non-parameteric Regression

$$\mathcal{V}_m^{NR}(\mathcal{C}) = \inf_{h_S = \mathcal{A}(\cdot)} \sup_{D_X, c \in \mathcal{C}} E_{S:|S|=m}\left[\left(h_S(x) - c(x)\right)^2\right]$$

Agnostic PAC-Learning

$$\mathcal{V}_m^{A-PAC}(\mathcal{C}) = \inf_{h_S = \mathcal{A}(\cdot)} \sup_{D_X, c \in \mathcal{C}} E_{S:|S|=m}\left[R(h_S) - \inf_{h' \in \mathcal{H}} R(h')\right]$$

Minimax Rates and Learning Tasks

PAC-Learning Classification

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Comparison of learning problems:

Case: $\mathcal{C} \subset \{\pm 1\}^{\mathcal{X}}$

$$4\mathcal{V}_m^{PAC}(\mathcal{C}) \leq \mathcal{V}_m^{NR}(\mathcal{C}) \leq \mathcal{V}_m^{A-PAC}(\mathcal{C})$$

Minimax Rates and Learning Tasks

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Statistical Learning Theory

Machine Learning Algorithms and Tasks

- **Guaranteed performance** for unknown distributions $D_{\mathcal{X}}$ requires we have some restriction on the hypothesis class $\mathcal{H}$ and concept class $\mathcal{C}$.

Image Classification

Abdellatif Abdelfattah Blog

Robotics / Controls

New Scientist
Delta, MIT and Cornell (Steven Collins)

Forecasting

washingtonpost.com

Statistical Learning Theory

Machine Learning Algorithms and Tasks

- **Guaranteed performance** for unknown distributions $D_{\mathcal{X}}$ requires we have some restriction on the hypothesis class $\mathcal{H}$ and concept class $\mathcal{C}$.
- There is **no general learning algorithm** that works for all possible tasks.
- These assertions correspond to so-called "**No Free Lunch Theorems.**"

Image Classification

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Robotics / Controls

New Scientist
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Statistical Learning Theory

Machine Learning Algorithms and Tasks

- **Guaranteed performance** for unknown distributions $D_{\mathcal{X}}$ requires we have some restriction on the hypothesis class $\mathcal{H}$ and concept class $\mathcal{C}$.
- There is **no general learning algorithm** that works for all possible tasks.
- These assertions correspond to so-called "**No Free Lunch Theorems.**"
- **To achieve good performance** learning algorithms must make some use of knowledge / mathematical structure of the specific task.

Image Classification

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Theorem: Let concept class be all binary functions, $\mathcal{C} = \mathcal{U}$ = {all functions $f(z): \mathcal{X} \rightarrow \{0,1\}$},
where $\mathcal{X}$ is discrete space of finite binary sequences $\{\{0,1\}^N, N \in \mathrm{N}\}$ = $\{(z_1, z_2, \ldots, z_N),\ z_i \in \{0,1\}\}$.
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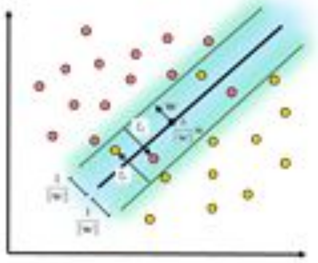
Support Vector Machines



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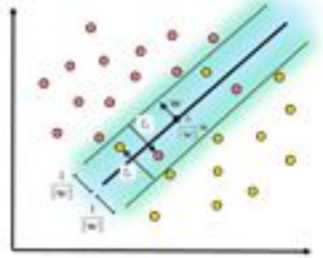
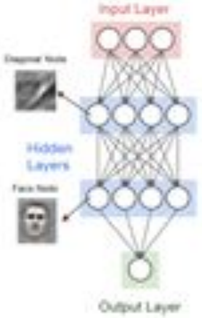


Neural Networks



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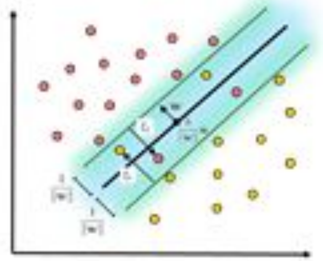

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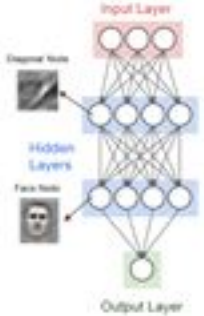
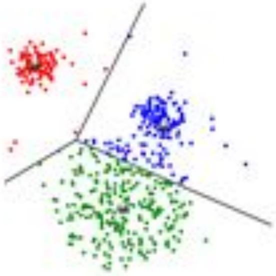


Clustering Methods



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