

Ritz-Galerkin Approximation

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206D: Finite Element Methods
University of California Santa Barbara

MATH 206D: Finite Element Methods

Welcome to MATH 206D: Finite Element Methods!

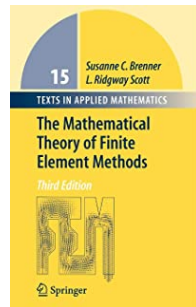
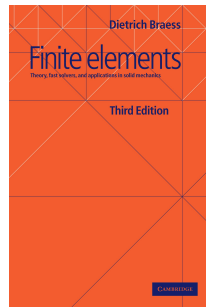
We will use the following books:

- *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics (third edition)*, D. Braess.
- *The Mathematical Theory of Finite Element Methods (third edition)*, S. Brenner and R. Scott.

For more information, see the course website:

<https://web.atzberger.org/teaching>

I look forward to working with you this quarter.



Introduction to Finite Element Methods

Variational Approach

Variational Principle

$$E[u] = \frac{1}{2} \int_0^1 (u'(x))^2 dx - \int_0^1 f(x)u(x)dx.$$

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We find that

$$\begin{aligned} (\delta E[u])(v) &= \int_0^1 u'(x) v'(x) dx - \int_0^1 v(x) f(x) dx \\ &= [u'(x) v(x)]_0^1 - \int_0^1 (u''(x) + f(x)) v(x) dx. \end{aligned}$$

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Suggests "natural boundary conditions" $\rightarrow u'(0) = u'(1) = 0$.

Variational Approach: Strong and Weak Forms

Variational Principle ($u'(0) = u'(1) = 0$)

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Implies PDE holds (strong form)

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We have "stiffness matrix" $[K]_{ij} = a(\phi_i, \phi_j)$ and "load vector" $[\mathbf{f}]_i = (f, \phi_i)$.

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Hence, if two solutions $u_{\mathcal{S}}$ and $\tilde{u}_{\mathcal{S}}$, then let $v = u_{\mathcal{S}} - \tilde{u}_{\mathcal{S}}$. We then have $a(v, \phi_i) = 0, \forall i$, so $v = 0 \Rightarrow u_{\mathcal{S}} = \tilde{u}_{\mathcal{S}}$ and $\text{Ker}\{K\} = 0$.



Shows the problem has a solution.

Ritz-Galerkin Approximation

Let $\mathcal{S} \subset \mathcal{V}$ where $\mathcal{S}$ is any finite dimensional subspace. Then we obtain numerical approximation as the $u_{\mathcal{S}}$ satisfying

$$a(u_{\mathcal{S}}, v) = (f, v), \quad v \in \mathcal{S}. \quad (\mathcal{P}_{\mathcal{S}})$$

The $u_{\mathcal{S}}$ provides the *Ritz-Galerkin Approximation* to solution u .

Theorem

Given $f \in L^2[0, 1]$ the problem $\mathcal{P}_{\mathcal{S}}$ has unique solution $u_{\mathcal{S}} \in \mathcal{S}$.

Proof (continued)

Hence, if two solutions $u_{\mathcal{S}}$ and $\tilde{u}_{\mathcal{S}}$, then let $v = u_{\mathcal{S}} - \tilde{u}_{\mathcal{S}}$. We then have $a(v, \phi_i) = 0, \forall i$, so $v = 0 \Rightarrow u_{\mathcal{S}} = \tilde{u}_{\mathcal{S}}$ and $\text{Ker}\{K\} = 0$.



Shows the problem has a solution.

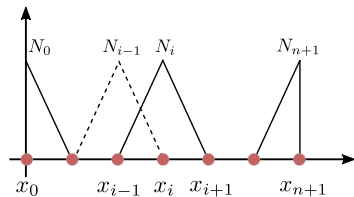
Still, need theory to show $u_{\mathcal{S}} \rightarrow u$ as $\mathcal{S} \rightarrow \mathcal{V}$ (i.e. we recover solution to the PDE in limit).

Linear Elements

Linear Elements

Consider space $\mathcal{S}$ generated by

$$v(x) = \sum_{i=0}^{n+1} v_i \phi_i(x)$$



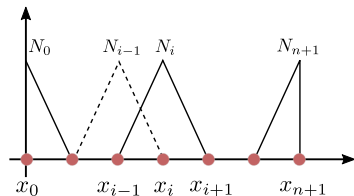
Linear Elements

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where $\phi_i(x) = N_i(x)$,

$$N_i(x) = \left\{ \begin{array}{ll} (x - x_{i-1})/h_i, & x \in [x_{i-1}, x_i] \\ (x_{i+1} - x)/h_{i+1}, & x \in [x_i, x_{i+1}] \\ 0, & \text{otherwise} \end{array} \right\}$$



(Hat Functions).

Linear Elements

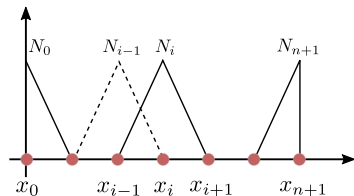
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Here, $h_i = x_i - x_{i-1}$ and $N_i(x_j) = \delta_{ij}$.



(Hat Functions).

Linear Elements

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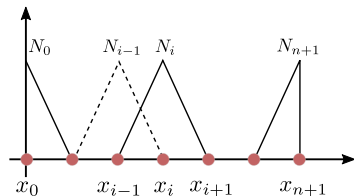
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Here, $h_i = x_i - x_{i-1}$ and $N_i(x_j) = \delta_{ij}$.

Mesh: $x_0, x_1, \dots, x_{n+1}$. **Elements:** $e_i = \{x | x_{i-1} \leq x \leq x_i\}$. **Shape Functions:** $N_i(x)$.



(Hat Functions).

Linear Elements

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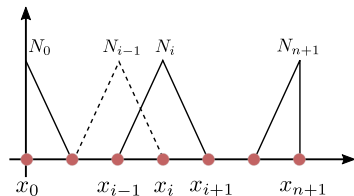
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Let $\mathcal{S} = \{v | v \in C[0, L], v(x) = \sum_{i=0}^{n+1} v_i N_i(x)\}$, referred to as the **shape space**.



(Hat Functions).

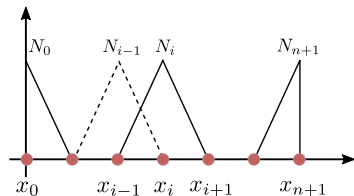
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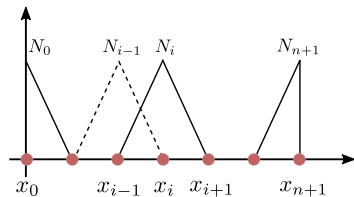
Let $\mathcal{S} = \{v | v \in C[0, L], v(x) = \sum_{i=0}^{n+1} v_i N_i(x)\}$, referred to as the **shape space**.

We would like to carry-out the Ritz-Galerkin approximations over this space.

Linear Elements

Shape functions:

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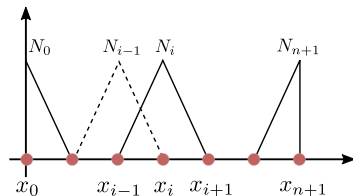
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Consider the heat equation in 1D on $[0, L]$

$$\left\{ \begin{array}{ll} \frac{d^2 u}{dx^2} = f(x), & x \in [0, L] \\ u(0) = T_1, u(L) = T_2, & x \text{ on boundary} \end{array} \right.$$



Linear Elements

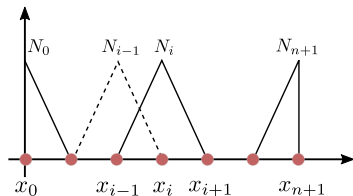
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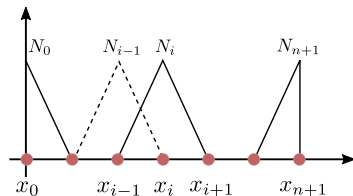
Boundary conditions: $u_0 = T_1$ and $u_{n+1} = T_2$ throughout.



Linear Elements

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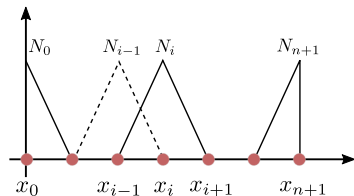
Boundary conditions: $u_0 = T_1$ and $u_{n+1} = T_2$ throughout. Weak form

$$a(u, v) = (f, v), \quad \forall v \in \mathcal{V}, \quad \text{WLOG } \mathcal{V} = \{v | v \in C[0, L], v(0) = v(L) = 0\}$$

Linear Elements

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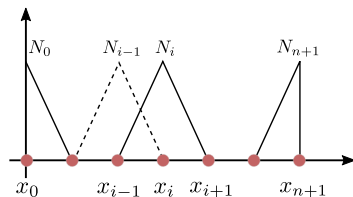
We obtain Ritz-Galerkin Approximation by considering finite dimensional problem

$$a(u_S, v) = (f, v), \quad \forall v \in \mathcal{S}, \quad \text{WLOG } \mathcal{S} = \{v | v = \sum_{i=1}^n v_i N_i(x)\}$$

Linear Elements

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To obtain stiffness matrix K and load vector $\mathbf{f}$, we need to compute the inner-products.

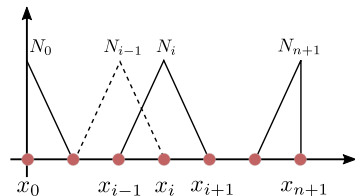
Linear Elements

Shape functions:

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Ritz-Galerkin Approximation

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Linear Elements

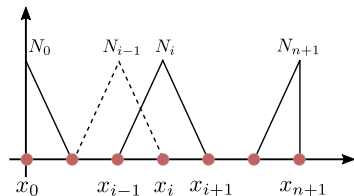
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Stiffness matrix $K_{ij} = a(N_i, N_j)$ when $|i - j| \leq 1$, $K_{ij} = 0$ otherwise.



Linear Elements

Shape functions:

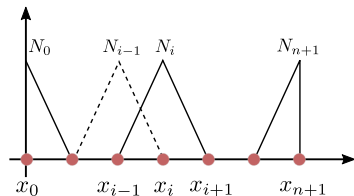
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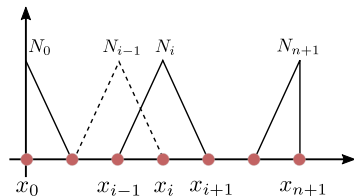
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Linear Elements

Shape functions:

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Ritz-Galerkin Approximation

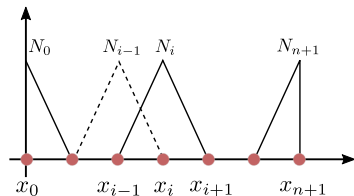
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$$(f, N_i) = \int_{x_{i-1}}^{x_{i+1}} f(x) N_i(x) dx.$$



Linear Elements

Shape functions:

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Ritz-Galerkin Approximation

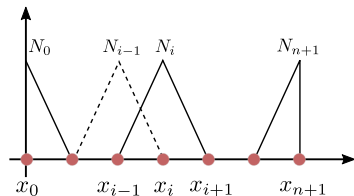
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When $f = \sum_{i=0}^{n+1} f_i N_i(x)$, compute via "mass matrix" $M_{ij} = (N_i, N_j)$, and $[f]_i = M_{ij} f_j$.

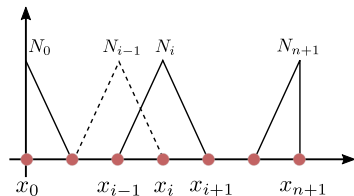
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Linear Elements

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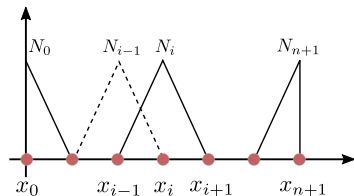
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Stiffness matrix when $h_i = h_0 = h$ and load vector when $f(x) = f_0$,

$$K = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ 0 & 0 & -1 & \ddots & -1 \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix},$$



Linear Elements

Shape functions:

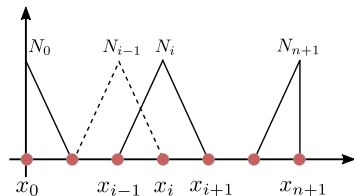
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Ritz-Galerkin Approximation

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Linear Elements

Shape functions:

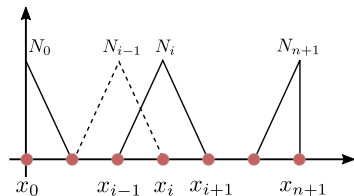
$$N_i(x) = \left\{ \begin{array}{ll} (x - x_{i-1})/h_i, & x \in [x_{i-1}, x_i] \\ (x_{i+1} - x)/h_{i+1}, & x \in [x_i, x_{i+1}] \\ 0, & \text{otherwise} \end{array} \right\}$$

Ritz-Galerkin Approximation

$$\left\{ \begin{array}{l} a(u_S, v) = (f, v), \forall v \in \mathcal{S} \\ \mathcal{S} = \{v | v = \sum_{i=0}^{n+1} v_i N_i(x)\} \end{array} \right.$$

Stiffness matrix when $h_i = h_0 = h$ and load vector when $f(x) = f_0$,

$$K = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ 0 & 0 & -1 & \ddots & -1 \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix}, \mathbf{f} = h \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n+1} \end{bmatrix}$$



In this case, the $K\mathbf{u} = \mathbf{f}$ has similarities to Finite Difference Method for the heat equation.

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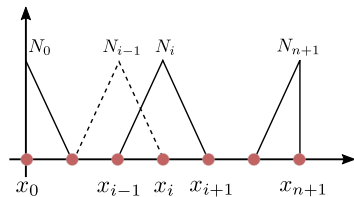
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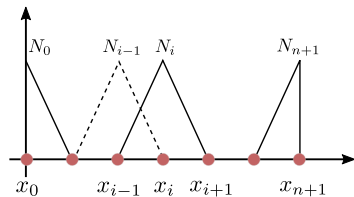
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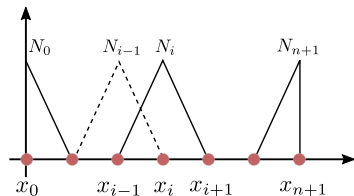
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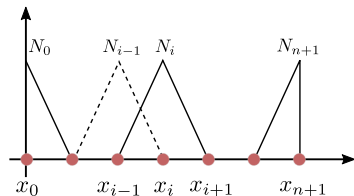
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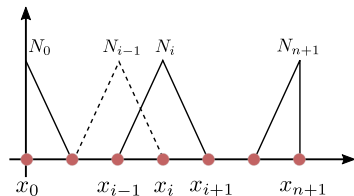
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Example (linear elements):

The Green's function for $d^2u/dx^2 = -f$ is given by

$$G(x, x_0) = \left\{ \begin{array}{ll} x, & x < x_0 \\ x_0, & x_0, \text{ otherwise} \end{array} \right\}, \quad \frac{dG}{dx} = \left\{ \begin{array}{ll} 1, & x < x_0 \\ 0, & x_0, \text{ otherwise} \end{array} \right\}, \quad \frac{d^2G}{dx^2} = -\delta(x - x_0).$$

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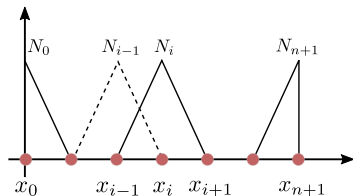
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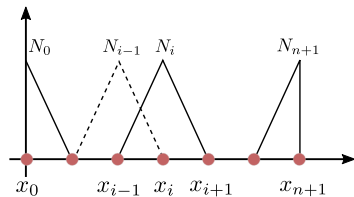
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The solution u above can be expressed as

$$u(x) = \int G(x, y) f(y) dx.$$

Error Estimates

Example (linear elements) (continued)

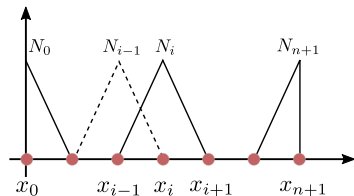


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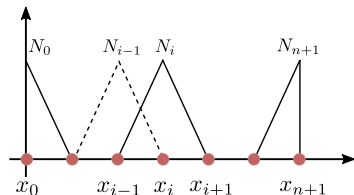
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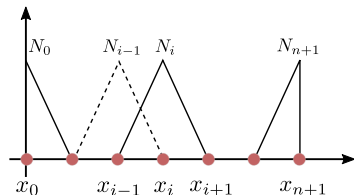
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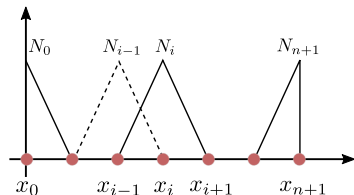
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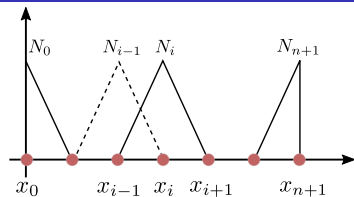
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This means u_S is piece-wise linear with $u_S(x_i) = u(x_i)$. We denote $u_S = u_I$ where u_I is the linear interpolation of the solution.



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 $\|u - u_I\|_\infty \leq Ch^2 \|u''\|_\infty$.

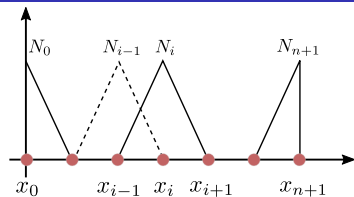


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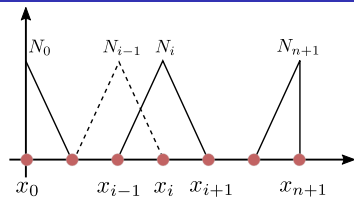
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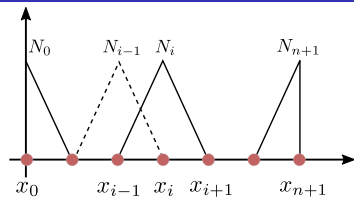
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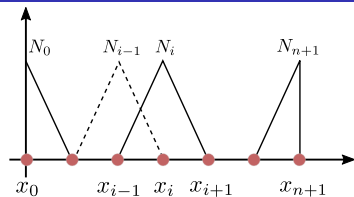
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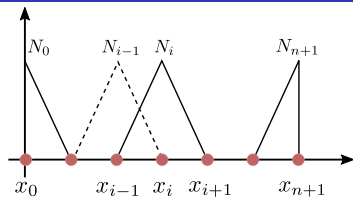
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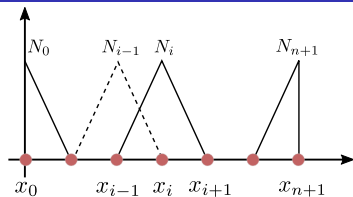
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Key is to design function spaces and study their interpolation theory, since this indicates FEM errors.

