Finite Element Spaces

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206D: Finite Element Methods
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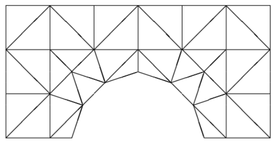
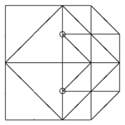


admissible triangulation



inadmissible (hanging nodes)

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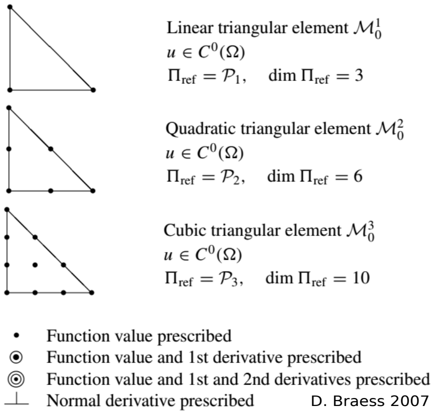

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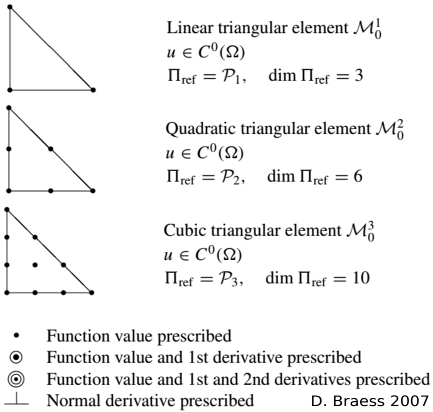

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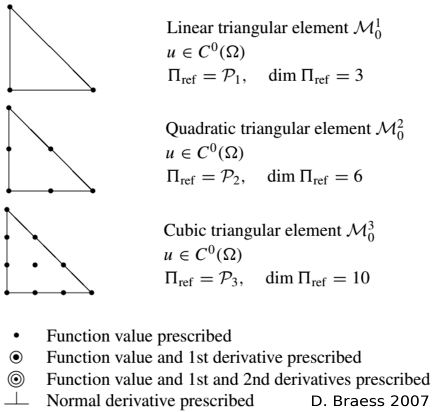

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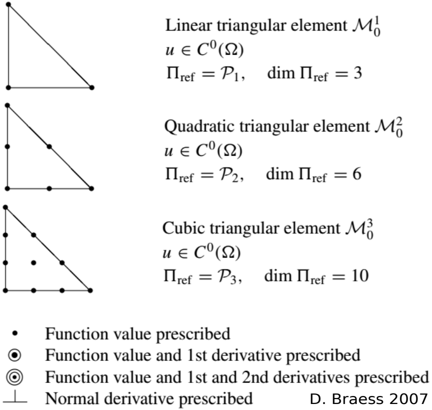

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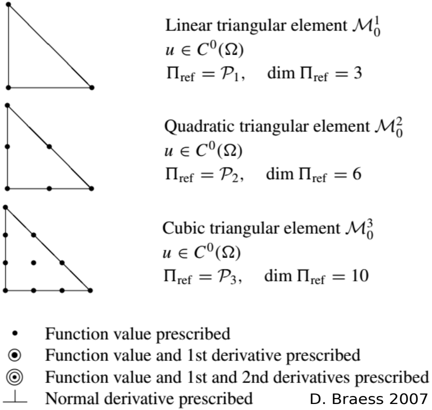

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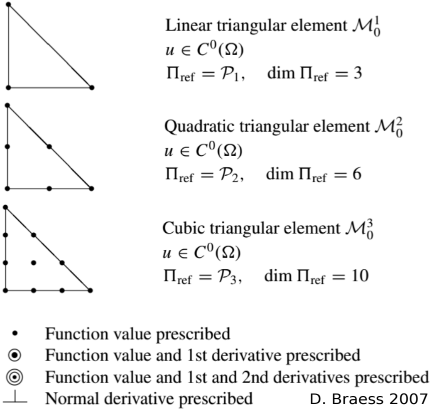

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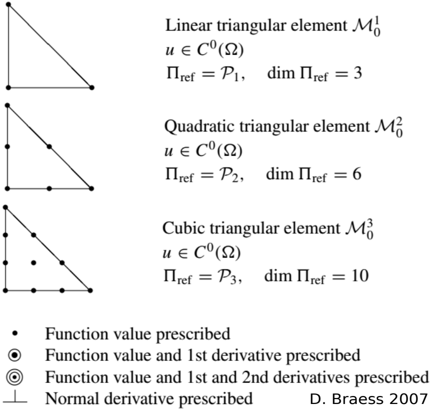

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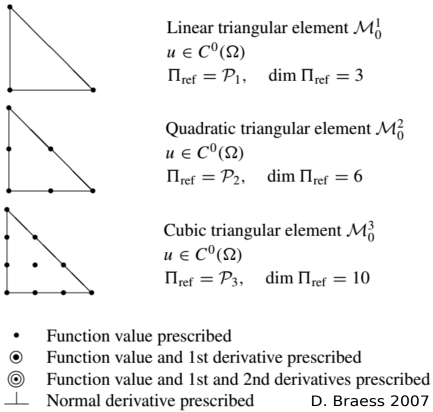

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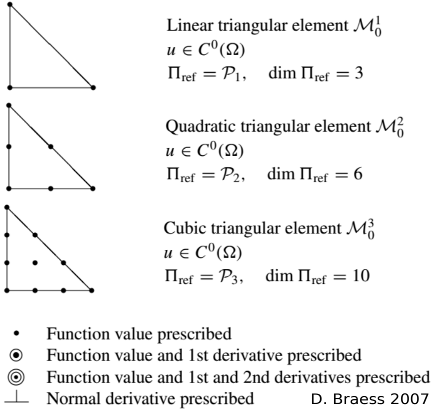

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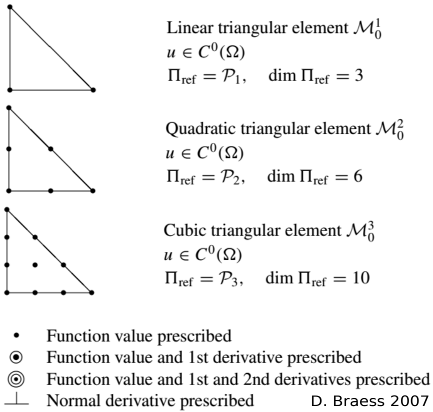

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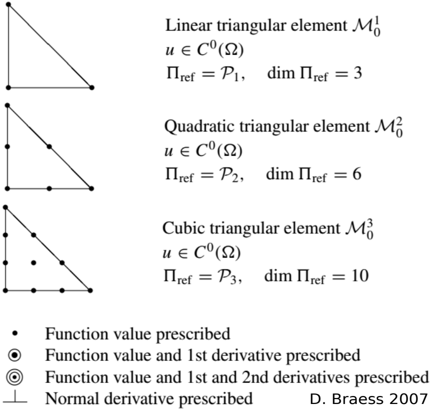

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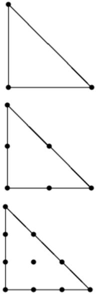

Linear triangular element $\mathcal{M}_0^1$
$u \in C^0(\Omega)$
$\Pi_{\text{ref}} = \mathcal{P}_1, \quad \dim \Pi_{\text{ref}} = 3$

Quadratic triangular element $\mathcal{M}_0^2$
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$u \in C^0(\Omega)$
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• Function value prescribed
⊙ Function value and 1st derivative prescribed
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Triangular Finite Elements: Lagrange Elements

Consider partition of the domain into triangular elements $\mathcal{T}$.

Lemma

Consider triangle T with $z_1, z_2, \ldots, z_s$, $s = 1 + 2 + \cdots (t+1)$ nodes lying on the lines depicted. For every $f \in C(T)$ there is a unique polynomial p of degree $\leq t$ satisfying interpolation

$$p(z_i) = f(z_i), \ \ 1 \leq i \leq s.$$

Proof: We proceed by induction. Clearly, in the case of $t = 0$ when $s = 1$ we have interpolation by the constant polynomials.

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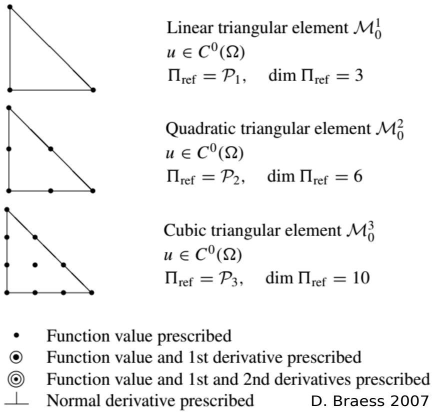

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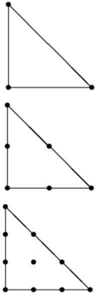

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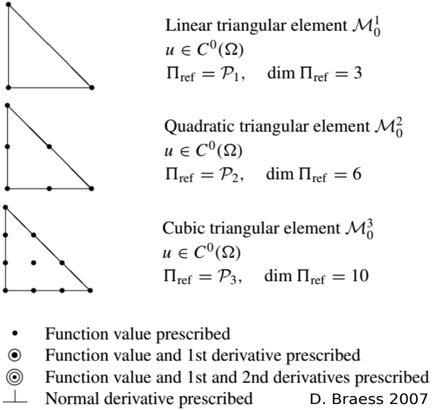

D. Braess 2007

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Consider partition of the domain into triangular elements $\mathcal{T}$.

Definition

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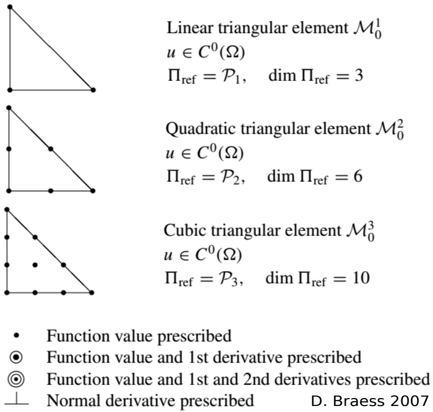

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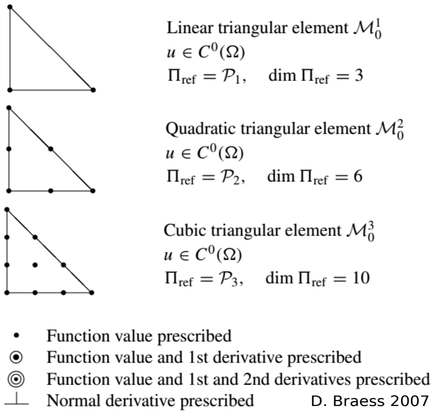

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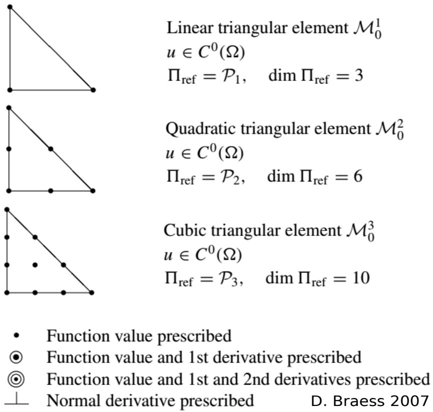

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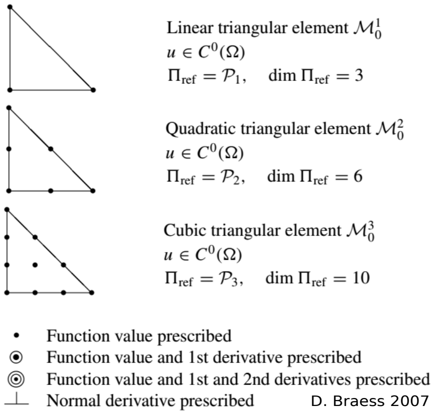

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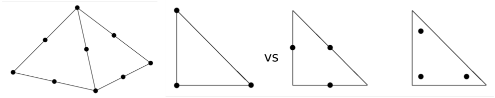

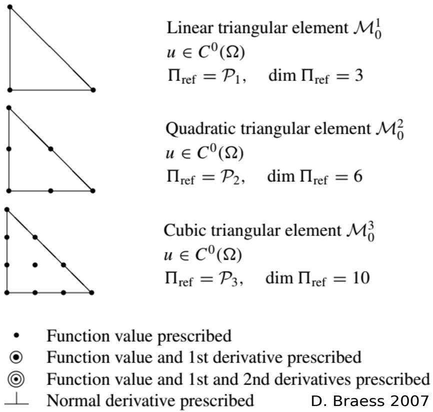

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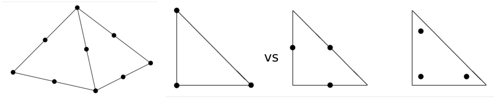

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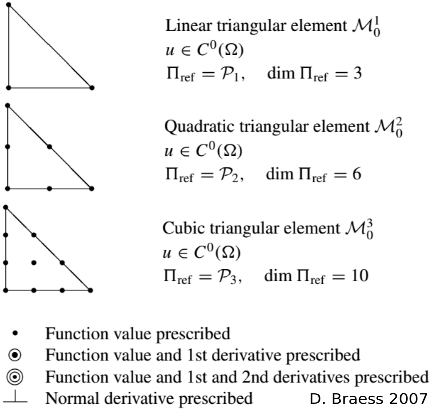

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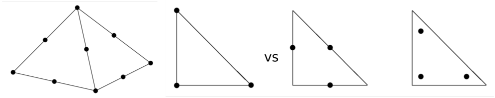

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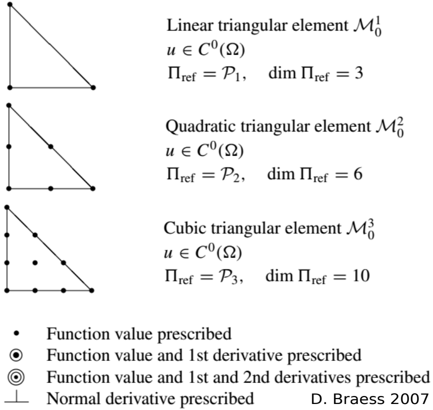
$\mathcal{M}_0^k$ is called the **conforming** P_k **element**.



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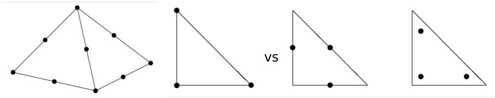

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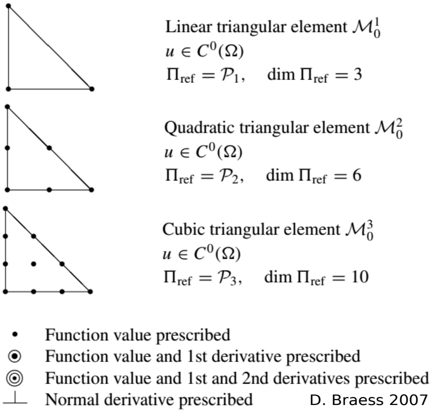

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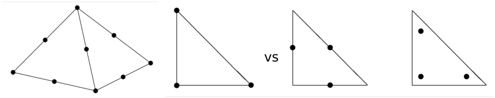

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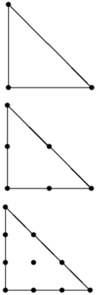
Nodal variables are $N_j(u) = u(z_j)$, so also called **Lagrange elements**.



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• Function value prescribed
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Triangular Finite Elements: C^1 Regularity

More challenging to obtain elements with C^1 regularity.

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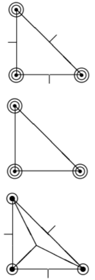

Argyris triangle
$u \in C^1(\Omega)$
$\Pi_{\mathrm{ref}} = \mathcal{P}_5, \quad \dim \Pi_{\mathrm{ref}} = 21$

Bell triangle
$u \in C^1(\Omega)$
$\Pi_{\mathrm{ref}} \subset \mathcal{P}_5, \ \partial_\nu u|_{\partial T_i} \in \mathcal{P}_3, \quad \dim \Pi_{\mathrm{ref}} = 18$

Hsieh–Clough–Tocher element
$u \in C^1(\Omega)$
$T = \bigcup_{i=1}^3 K_i, \quad u|_{K_i} \in \mathcal{P}_3, \quad \dim \Pi_{\mathrm{ref}} = 12$

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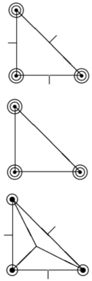

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Triangular Finite Elements: C^1 Regularity

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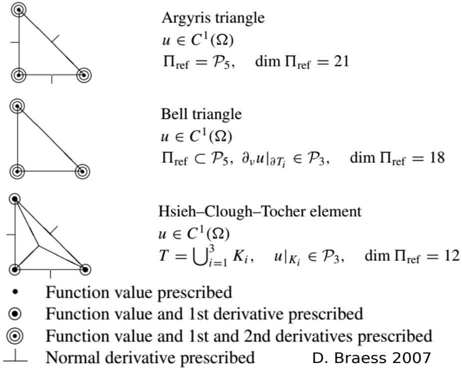
Argyris element:
Uses $\mathcal{P}_5$ which has $\dim \mathcal{P}_5 = 21$.



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Triangular Finite Elements: C^1 Regularity

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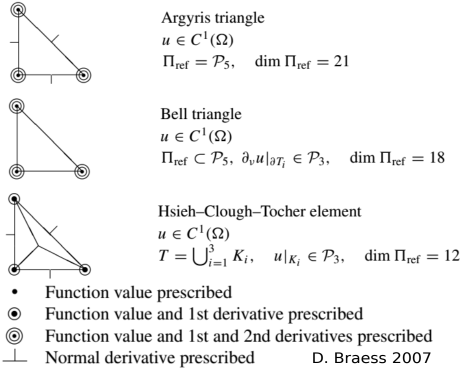
Argyris element:
Uses $\mathcal{P}_5$ which has $\dim \mathcal{P}_5 = 21$.
Values given of all derivatives up to order 2 at the vertices.



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Uses $\mathcal{P}_5$ which has $\dim \mathcal{P}_5 = 21$.
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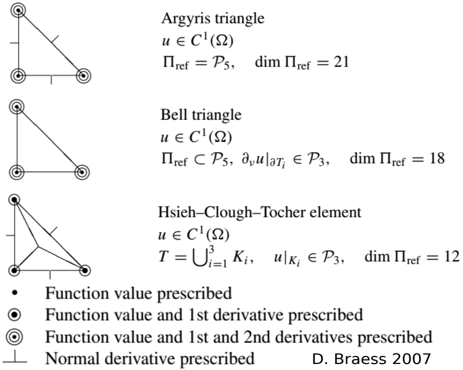
However, this is only $3 \times 6 = 18$ DOF.



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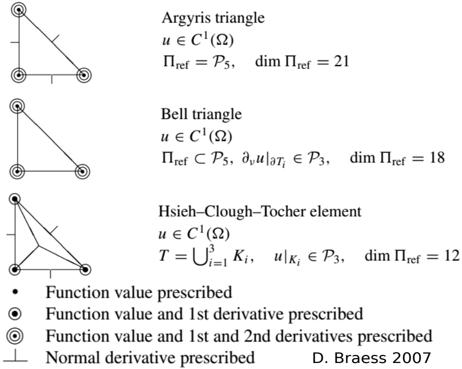

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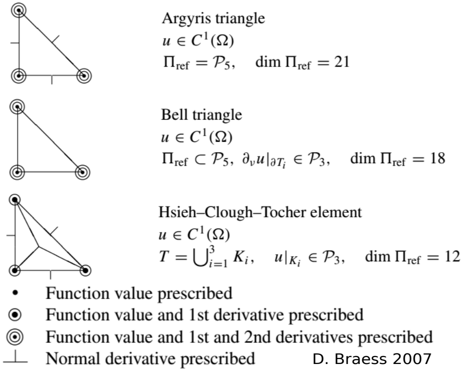

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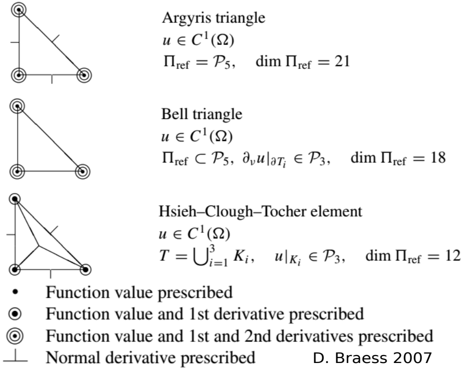

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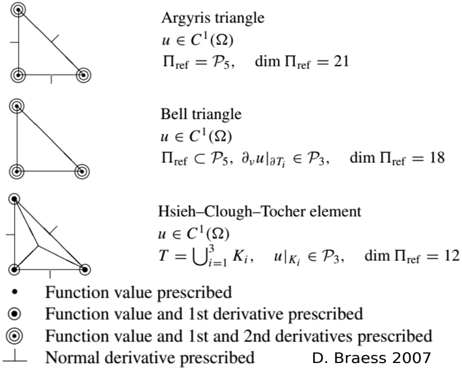

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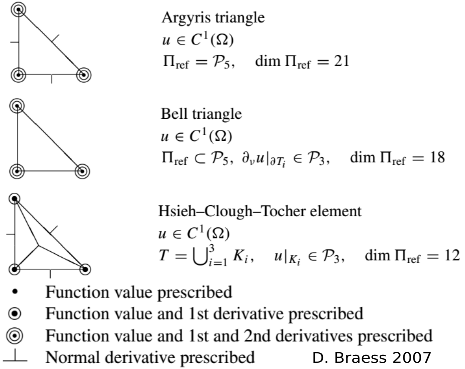

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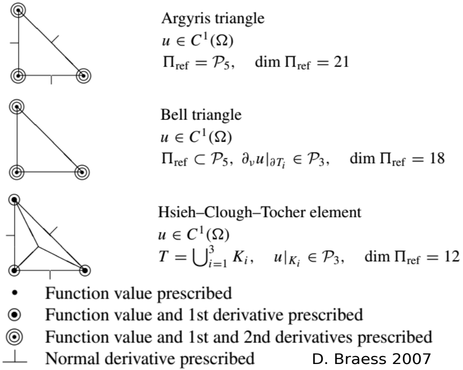

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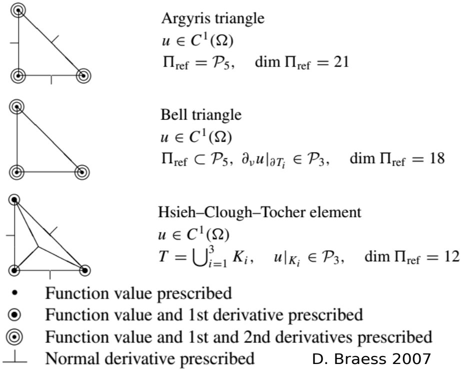
Hsieh-Clough-Tocher element: Macroelement approach.



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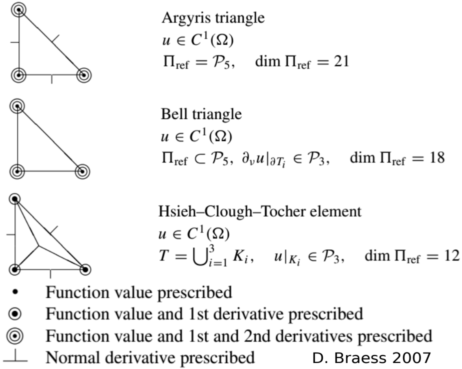
Subdivide the triangle into three subtriangles.



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Triangular Finite Elements: C^1 Regularity

More challenging to obtain elements with C^1 regularity.

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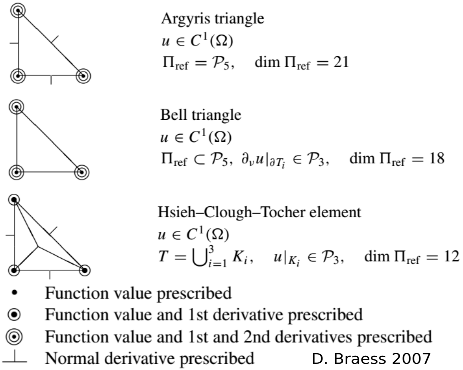

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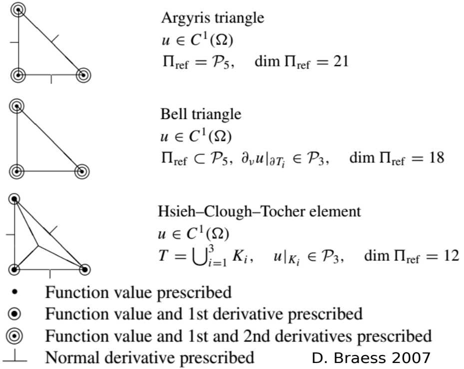

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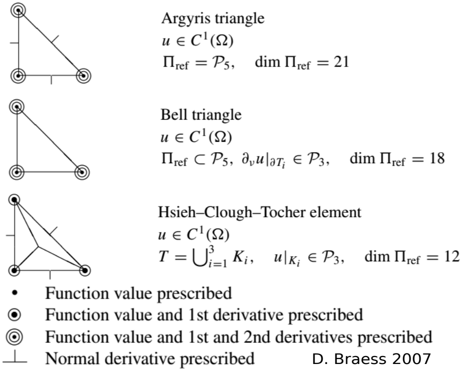

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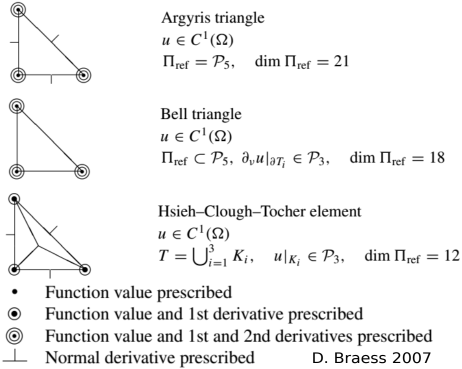

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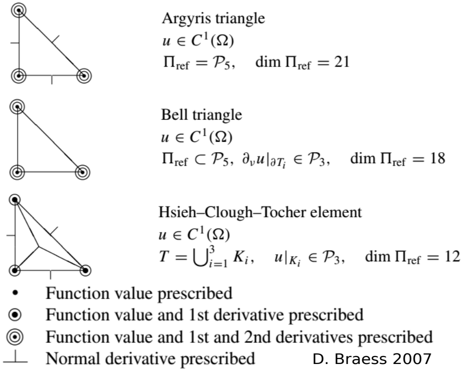

Bernstein-Bézier representation of polynomials used to handle derivatives along element boundaries.

Quadrilateral Finite Elements

Tensor Product Bases

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A **tensor-product basis** generated by $\{\phi_k\}_{k=1}^{t}$ for $\mathbf{x} \in \mathbb{R}^n$

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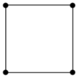

Bilinear quadrilateral element Q_1
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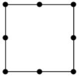

Serendipity element
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Quadrilateral Finite Elements

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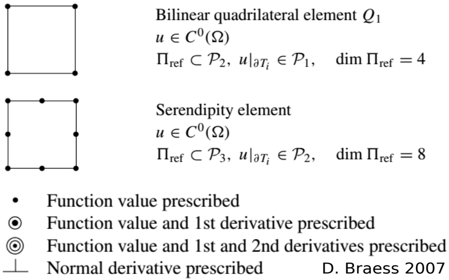

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Quadrilateral Finite Elements

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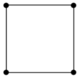

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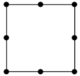

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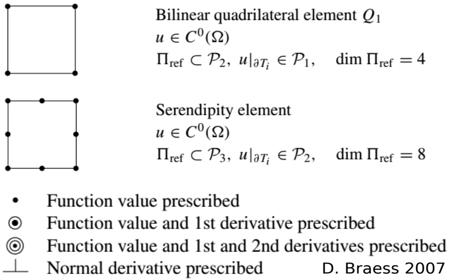

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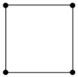
Quadrilateral Finite Elements



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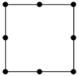

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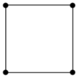

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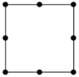

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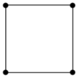

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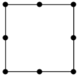

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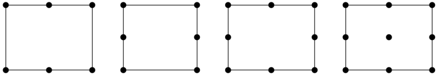

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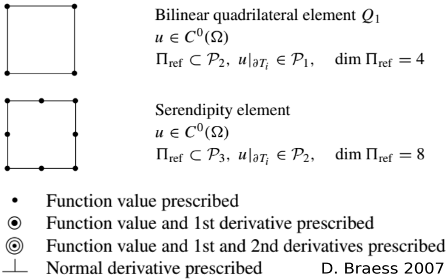

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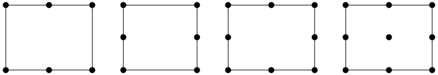

Quadrilateral Finite Elements

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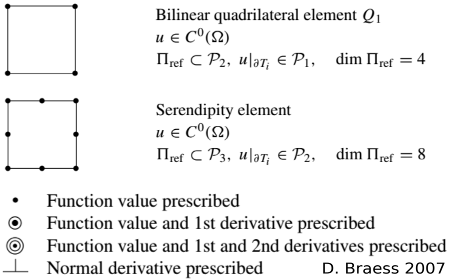
$$\begin{aligned} p(x,y) = {} & c_0 + c_1 x + c_2 y + c_3 xy \\ & + c_4(x^2-1)(y-1) + c_5(x^2-1)(y+1) \\ & + c_6(x-1)(y^2-1) + c_7(x+1)(y^2-1). \end{aligned}$$



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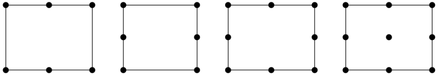

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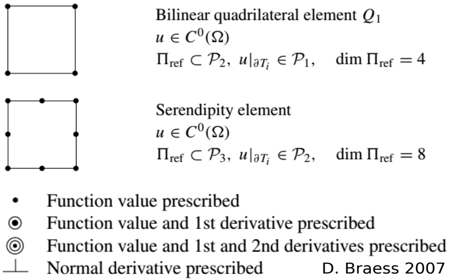
Nodal locations are vertices of rectangle and edge mid-points.



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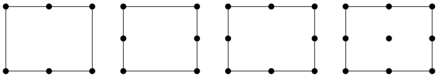

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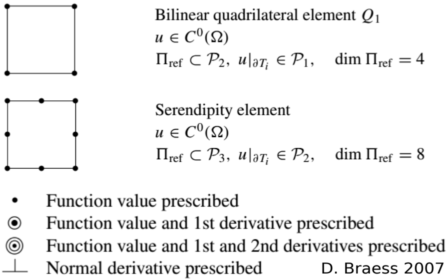

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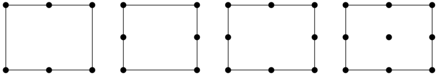

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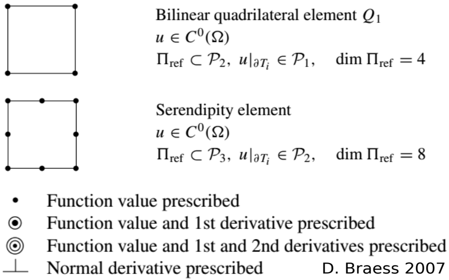
9-**Point Element:**

Consider $\mathcal{S}_9 = \mathcal{S}_{sd} \bigoplus \{c_8(x^2-1)(y^2-1)\}$.



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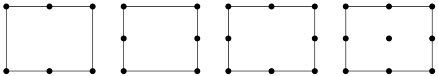

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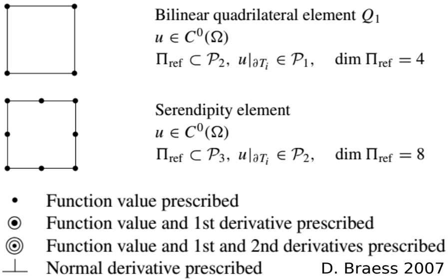

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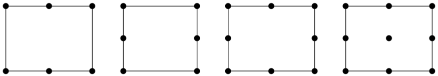

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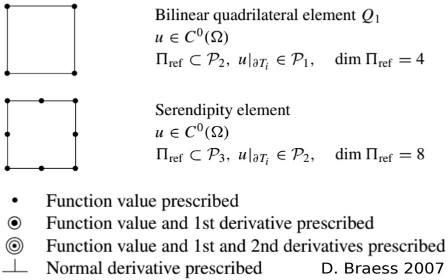

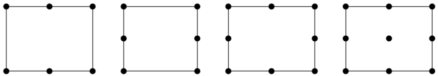

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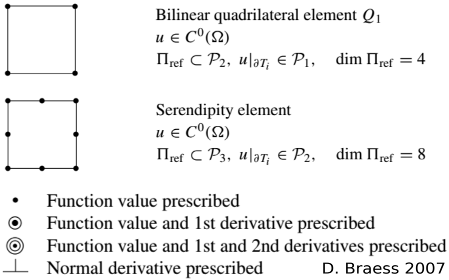

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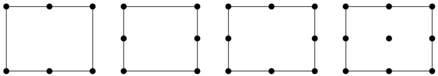

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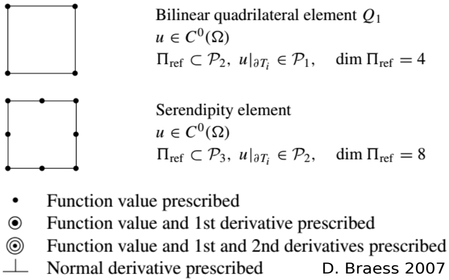

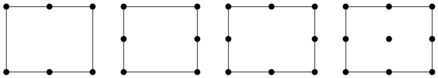

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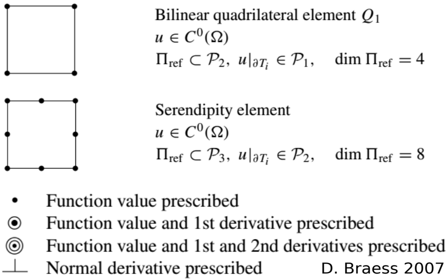

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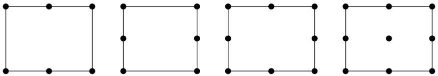

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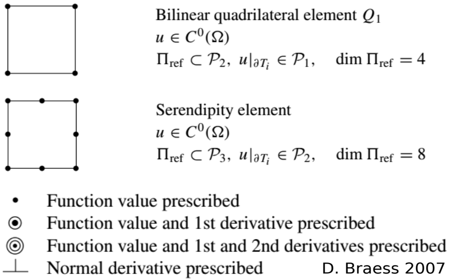

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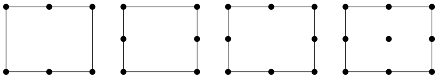

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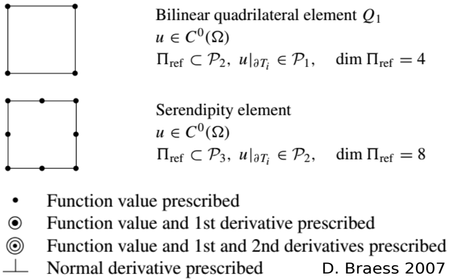

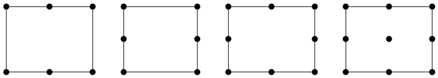

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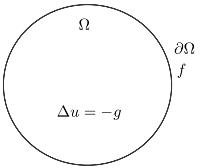

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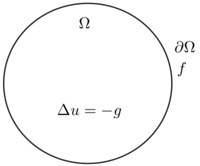

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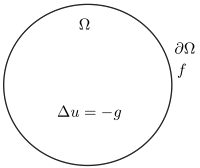

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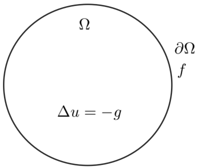

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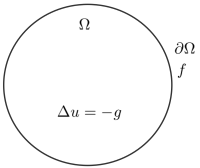

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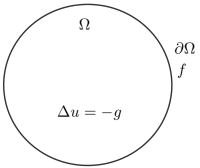

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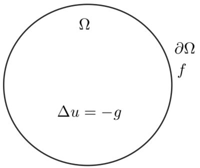

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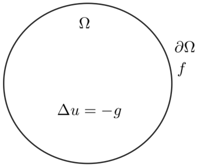

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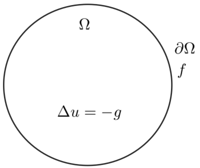

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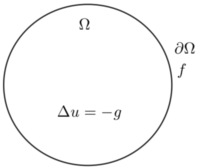

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Consider Poisson equation with Dirichlet boundary conditions and Ritz-Galerkin (RG) Approximation:

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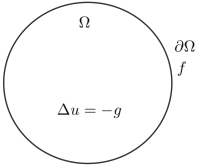

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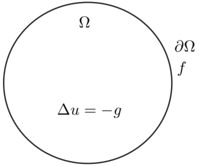
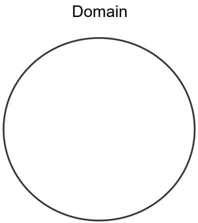


Discretization:

Divide domain into triangular elements T_j.



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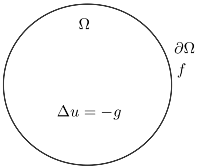

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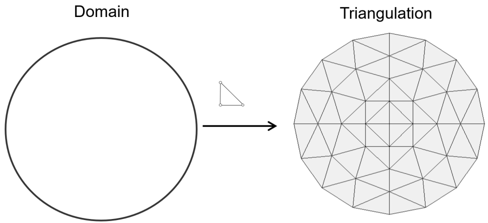

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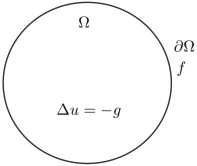

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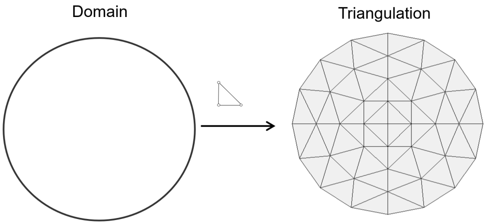

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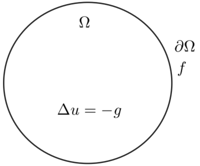

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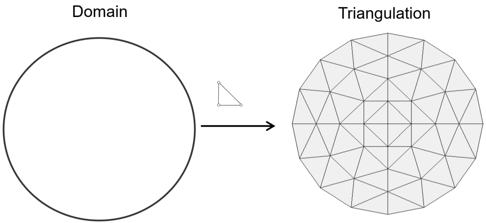
Use for shape space $\mathcal{P}_1$.



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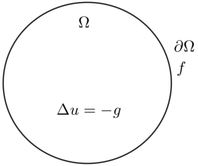

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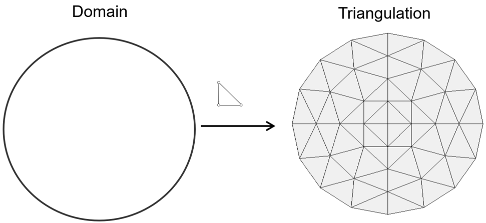

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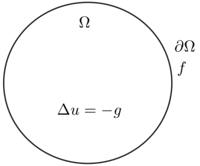

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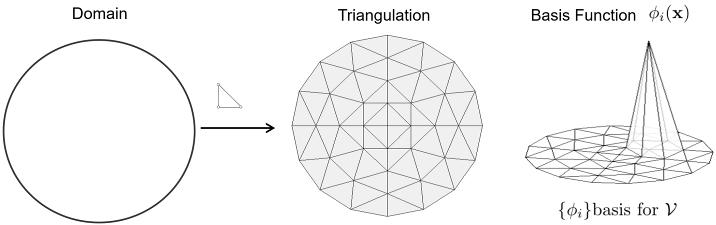

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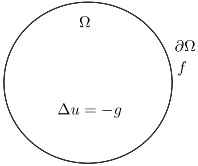

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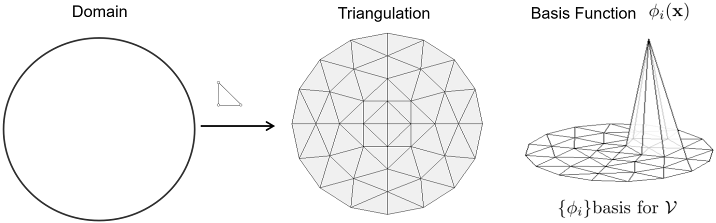
Nodal basis $\{\phi_i\}$ are 2D "hat functions."



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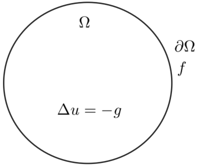

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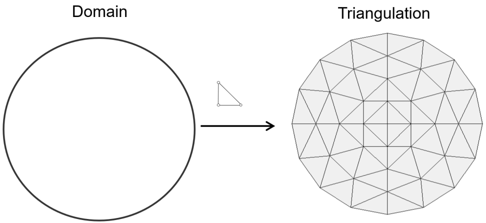
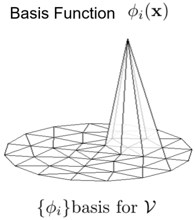
Functions in $v \in \mathcal{S}$ can be represented as

$$v(x) = \sum_{i=1}^{n} v(\mathbf{x}_i)\phi_i(\mathbf{x}) \in H^1.$$





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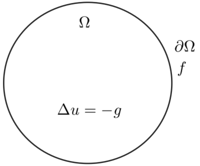

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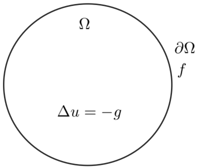

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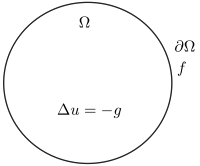

Mesh Refinement:

Can increase accuracy by refining the mesh.

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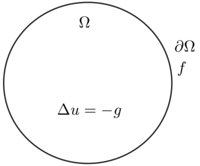

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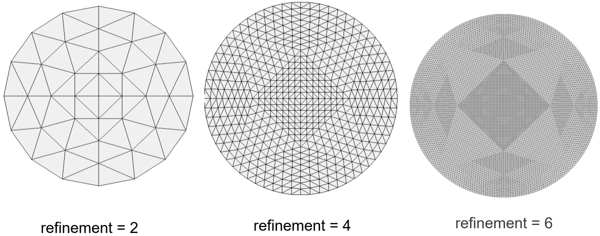

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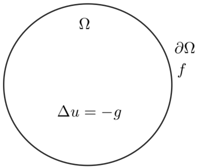

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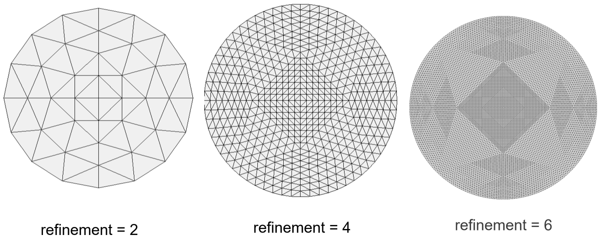

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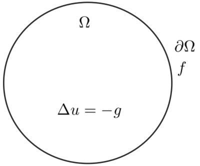

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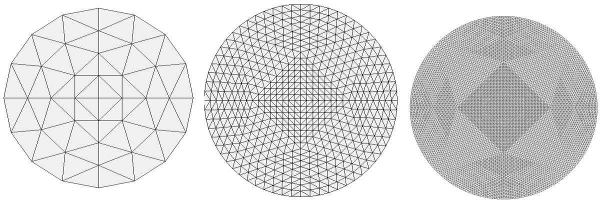

refinement = 2 refinement = 4 refinement = 6

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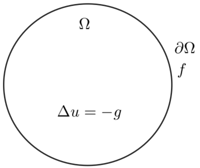

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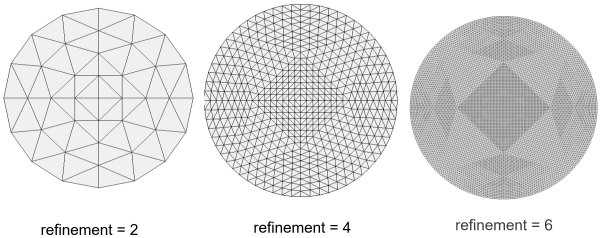

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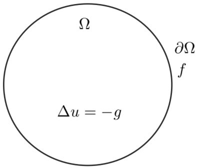

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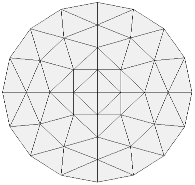
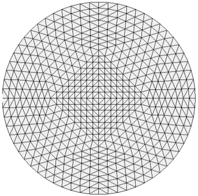
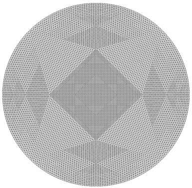


refinement = 2



refinement = 4



refinement = 6

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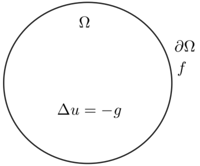

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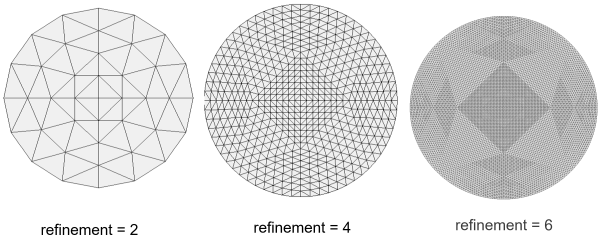
Quality of the triangle shapes is important.

Quality impacts condition number of the stiffness matrix K.



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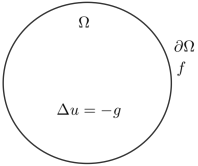

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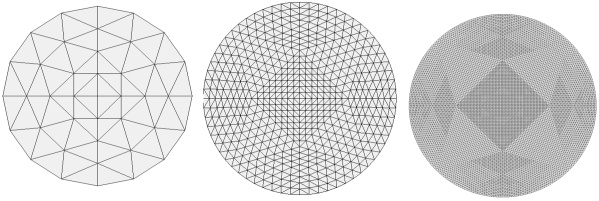
Convergence expected sufficiently uniform refinements.



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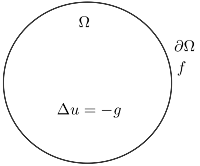

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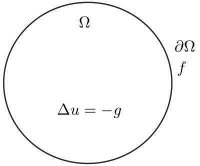

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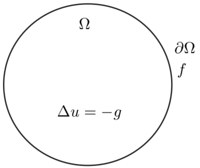

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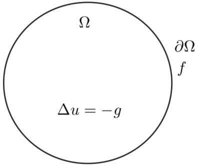

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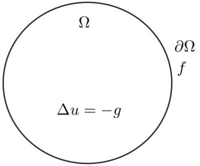

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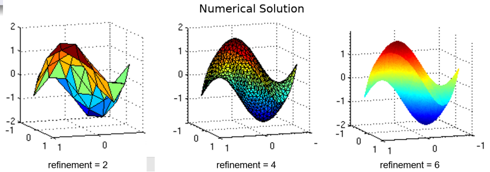

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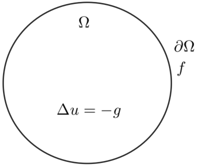

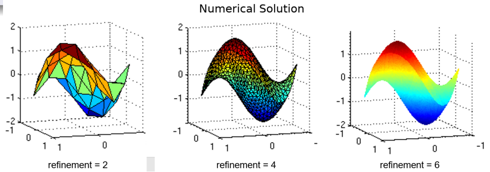

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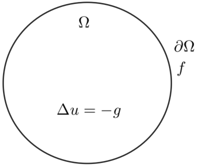

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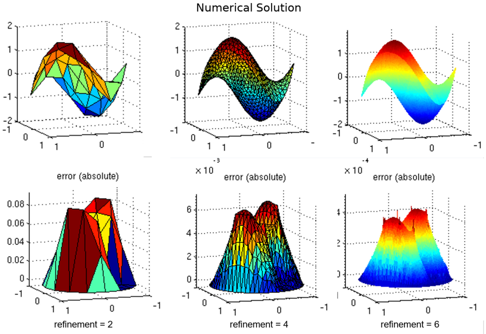

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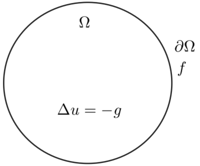

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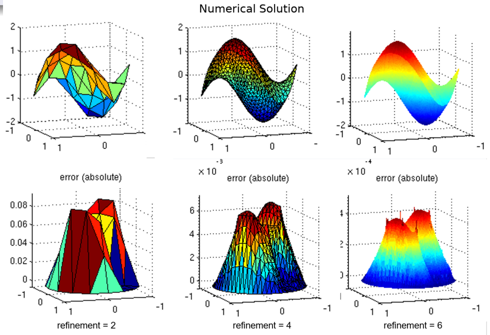

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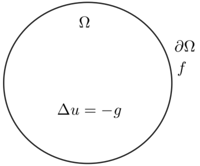

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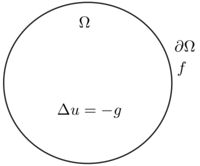

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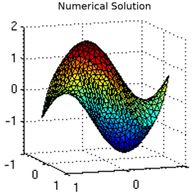

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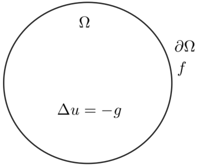

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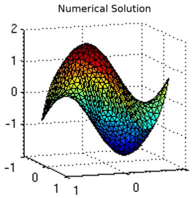

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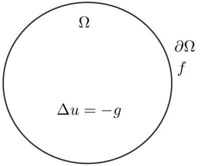

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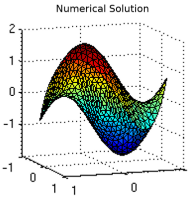

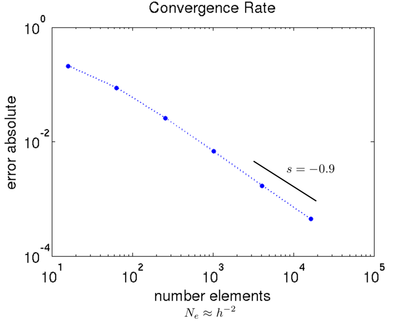

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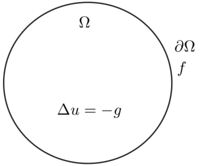

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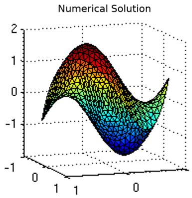
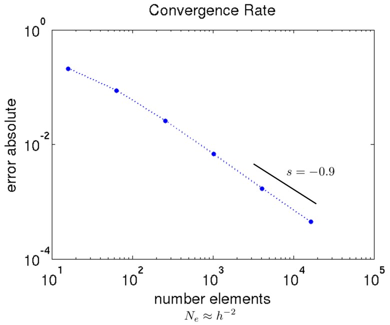
Log-log plots yield information on convergence rate





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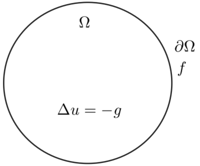

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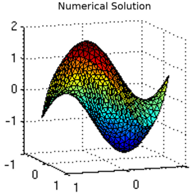
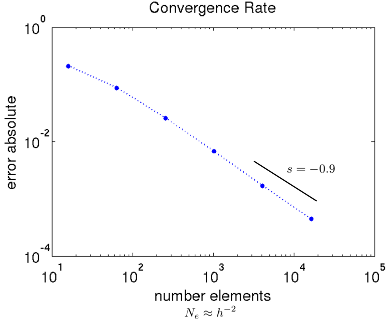
Log-log plots yield information on convergence rate $\epsilon = Ch^r$





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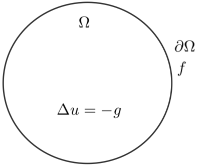

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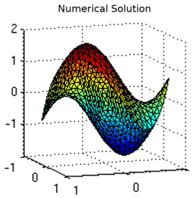

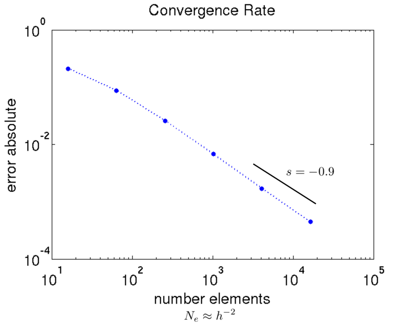

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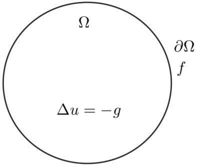

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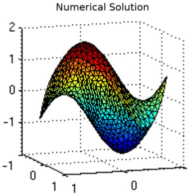
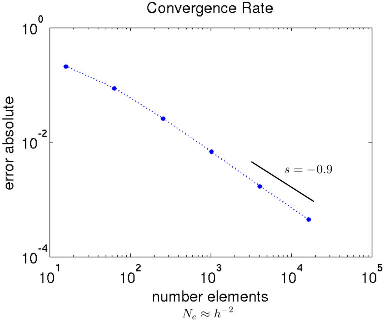
$\epsilon = Ch^r \rightarrow \log(\epsilon) = \log(h)r + \log(C) \Rightarrow \; -r/2 = s \sim -0.9$





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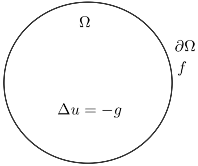

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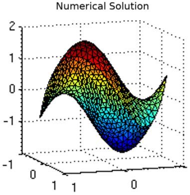

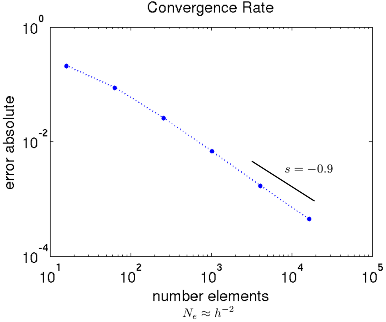

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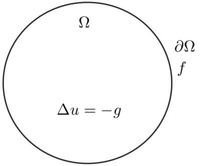

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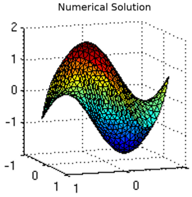
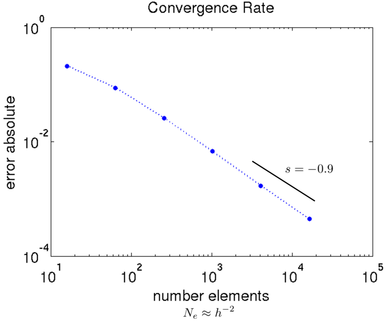




Log-log plots yield information on convergence rate

$\epsilon = Ch^r \rightarrow \log(\epsilon) = \log(h)r + \log(C) \Rightarrow -r/2 = s \sim -0.9 \rightarrow r \sim 1.8.$

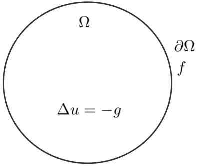
Indicates 2^{nd}-order convergence rate.

Application to Elliptic PDEs

Poisson Equation as Model Problem :

Consider Poisson equation with Dirichlet boundary conditions and Ritz-Galerkin (RG) Approximation:

$$\left\{ \begin{array}{ll} \Delta u = -g, & x \in \Omega \\ u = f, & x \in \partial\Omega. \end{array} \right\} \rightarrow \left\{ \begin{array}{l} a(u,v) = (g,v), \;\; v \in \mathcal{S} \\ a(u,v) = \int_\Omega \nabla u \cdot \nabla v d\mathbf{x}. \end{array} \right\} \text{ (RG-Approximation)}$$



Example:

Consider PDE with

$$g(x,y) = \pi^2 \sin(\pi x) + \pi^2 \cos(\pi x)$$
$$f(x,y) = \sin(\pi x) + \cos(\pi x).$$

Solution is

$$u(x,y) = \sin(\pi x) + \cos(\pi x).$$

Study the error vs mesh refinement $N \sim h^{-2}$.

Log-log plots yield information on convergence rate

$\epsilon = Ch^r \rightarrow \log(\epsilon) = \log(h)r + \log(C) \Rightarrow -r/2 = s \sim -0.9 \rightarrow r \sim 1.8.$

Indicates 2^{nd}-order convergence rate.